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INTEGRAL SURFACES OF PAIRS OF DIFFERENTIAL EQUATIONS OF THE THIRD ORDER

A DISSERTATION SUBMITTED TO THE
GRADUATE FACULTY IN CANDIDACY FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

By
CHARLES FRANKLIN BOWLES

L. C.

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SECTION I

INTRODUCTION

We begin by recalling the definition of a surface and explaining what is meant by an integral surface of a system of differential equations. In a linear space of n dimensions, which we shall denote by S_n, a point P_x has n+1 projective, homogeneous coordinates $x^{(1)}$, ---, $x^{(n+1)}$. If these coordinates are functions of two independent variables u and v, so that

$$x^{(i)} = x^{(i)}(u,v), \qquad (i = 1, \cdots, n+1),$$

or in vector notation,

$$x = x(u,v),$$

then the locus of P_x, as u and v vary, is a surface S in the space S_n. If all n+1 of the coordinates of P_x are solutions of a system of one or more linear, homogeneous, partial, differential equations in the dependent variable x and the independent variables u and v, then the surface S is said to be an integral surface of the system of differential equations.

In a recent paper, Lane has considered the integral surfaces of a pair of differential equations of the third order,[1] which he wrote in the form

---

[1] E. P. Lane, Integral Surfaces of Pairs of Partial Differential Equations of the Third Order, Transactions of the American Mathematical Society, vol. 32 (1930).

$$
\text{(I)}\qquad
\begin{aligned}
&A x_{uuu} + 3B x_{uuv} + 3C x_{uvv} + D x_{vvv} + T = 0,\\
&A' x_{uuu} + 3B' x_{uuv} + 3C' x_{uvv} + D' x_{vvv} + T' = 0,
\end{aligned}
$$

where T and T' denote linear combinations of the derivatives of x of the second and lower orders, including x undifferentiated, and where the coefficients are scalar functions of u and v not all identically zero. He assumed that the two equations are essentially distinct in the sense that the coefficients of the derivatives of the third order are not proportional, and that an integral surface of this pair of equations is not an integral surface of more than two such essentially distinct equations of the third order, and is not an integral surface of any equation of order lower than three. By considering certain covariantly defined triples of one-parameter families of curves on the integral surfaces he showed that a pair of equations of form (I) can always be reduced to one of six canonical forms, and that the integral surfaces of these six canonical forms are geometrically distinct, besides making certain further observations concerning the geometry of these surfaces, particularly with reference to integral surfaces of the first four canonical forms.

It is the purpose of this paper to continue this study of the integral surfaces of a pair of differential equations of the third order. We shall first calculate the integrability conditions for each of the six canonical forms, and then shall restrict our attention to the study of the geometry of the integral surfaces of the fifth and sixth of these forms. In Section III we shall consider transformations of these two canoni-

cal forms, paying particular attention to those transformations which preserve the forms of the equations. In Section IV we shall set up for each integral surface a system of local coordinates by means of which we shall study the properties of systems of hyperplanes that have with the surface contacts of various orders. In Section V we shall consider in detail the osculating spaces to curves on the integral surfaces, and their loci as the curves vary in a prescribed manner. Finally in Section VI we shall digress from the study of the integral surfaces long enough to develop some extensions of Bompiani's theory of quasi-asymptotic curves upon a general surface, and then shall apply this theory in determining the quasi-asymptotics upon our integral surfaces.

SECTION II

INTEGRABILITY CONDITIONS

In order that a given pair of differential equations be completely integrable its coefficients must satisfy certain partial differential equations known as integrability conditions. These conditions are found by computing the derivatives of x of higher orders, and in case a derivative can be computed in two ways requiring that the two computed values be equal. In this section we shall compute the integrability conditions for each of the six canonical forms to which a pair of differential equations of the third order may be reduced.

1. The First Canonical Form. The equations of the first canonical form are

$$(1)\qquad \begin{aligned} x_{uuu} &= a x_{uu} + h x_{uv} + b x_{vv} + l x_u + m x_v + d x, \\ x_{vvv} &= a' x_{uu} + h' x_{uv} + b' x_{vv} + l' x_u + m' x_v + d' x. \end{aligned}$$

In order to simplify the computation we note that the equations are symmetrical under the substitution

$$(2)\qquad \begin{pmatrix} u & a & h & b & l & m & d \\ v & b' & h' & a' & m' & l' & d' \end{pmatrix}$$

That is, if each letter in (1) be replaced by the one above or below it in (2) the equations (1) will be reproduced.

Differentiating each equation of (1) with respect to u and v we find

$$
\begin{aligned}
x_{uuuu} &= \alpha_1 x_{uuv} + \alpha_2 x_{uvv} + \alpha_3 x_{uu} + \alpha_4 x_{uv} + \alpha_5 x_{vv} \\
&\quad + \alpha_6 x_u + \alpha_7 x_v + \alpha_8 x, \\
x_{uuuv} &= \beta_1 x_{uuv} + \beta_2 x_{uvv} + \beta_3 x_{uu} + \beta_4 x_{uv} + \beta_5 x_{vv} \\
&\quad + \beta_6 x_u + \beta_7 x_v + \beta_8 x, \\
x_{uvvv} &= \delta_1 x_{uuv} + \delta_2 x_{uvv} + \delta_3 x_{uu} + \delta_4 x_{uv} + \delta_5 x_{vv} \\
&\quad + \delta_6 x_u + \delta_7 x_v + \delta_8 x, \\
x_{vvvv} &= \epsilon_1 x_{uuv} + \epsilon_2 x_{uvv} + \epsilon_3 x_{uu} + \epsilon_4 x_{uv} + \epsilon_5 x_{vv} \\
&\quad + \epsilon_6 x_u + \epsilon_7 x_v + \epsilon_8 x,
\end{aligned} \tag{3}
$$

where

$$
\begin{aligned}
&\alpha_1 = k, \quad \alpha_2 = b, \quad \alpha_3 = a^2 + a_u + l, \\
&\alpha_4 = ak + k_u + m, \quad \alpha_5 = ab + b_u, \\
&\alpha_6 = al + l_u + d, \quad \alpha_7 = am + m_u, \\
&\alpha_8 = ad + d_u \\
&\beta_1 = a, \quad \beta_2 = k, \quad \beta_3 = ba' + a_v, \\
&\beta_4 = bk' + k_v + l, \quad \beta_5 = bb' + b_v + m, \\
&\beta_6 = bl' + l_v, \quad \beta_7 = bm' + m_v + d, \\
&\beta_8 = bd' + d_v.
\end{aligned} \tag{4}
$$

The values for the δ's and the ϵ's are obtained from those of the α's and the β's by means of the substitution (2) augmented by

$$
\begin{pmatrix} \alpha & \beta & 1 & 3 & 6 \\ \epsilon & \delta & 2 & 5 & 7 \end{pmatrix}, \tag{5}
$$

the subscripts 4 and 8 remaining unchanged.

Differentiating each equation of (3) with respect to u and v we find, on reducing the coefficients by means of (4),

that the expressions for $(x_{uuuu})_v$ and $(x_{uuuv})_u$ are identical. Hence by the symmetry of the substitution the expressions for $(x_{uvvv})_v$ and $(x_{vvvv})_u$ are identical. The values for x_{uuuvv} and x_{uuvvv} which we shall need in the later computation are

$$(6) \qquad x_{uuuvv} = a\,x_{uuvvv} + C_1 x_{uuvv} + C_2 x_{uvvv} + C_3 x_{uu} + C_4 x_{uv} + C_5 x_{vv} + C_6 x_u + C_7 x_v + C_8 x,$$

$$(7) \qquad x_{uuvvv} = b'x_{uuvvv} + D_1 x_{uuvv} + D_2 x_{uvvv} + D_3 x_{uu} + D_4 x_{uv} + D_5 x_{vv} + D_6 x_u + D_7 x_v + D_8 x,$$

where

$$C_1 = \beta_2\delta_1 + \beta_{1v} + \beta_3, \qquad C_2 = \beta_2\delta_2 + \beta_{2v} + \beta_4,$$
$$C_3 = \beta_2\delta_3 + \beta_{3v} + a'\beta_5, \qquad C_4 = \beta_2\delta_4 + \beta_{4v} + b'\beta_5 + \beta_6,$$
$$C_5 = \beta_2\delta_5 + \beta_{5v} + b'\beta_5 + \beta_7, \qquad C_6 = \beta_2\delta_6 + \beta_{6v} + l'\beta_5,$$
$$C_7 = \beta_2\delta_7 + \beta_{7v} + m'\beta_5 + \beta_8, \qquad C_8 = \beta_2\delta_8 + \beta_{8v} + d'\beta_5,$$
$$D_1 = \delta_1\beta_1 + \delta_{1u} + \delta_4, \qquad D_2 = \delta_1\beta_2 + \delta_{2u} + \delta_5$$
$$D_3 = \delta_1\beta_3 + \delta_{3u} + a\,\delta_5 + \delta_6, \qquad D_4 = \delta_1\beta_4 + \delta_{4u} + b\,\delta_5 + \delta_7,$$
$$D_5 = \delta_1\beta_5 + \delta_{5u} + b\,\delta_5, \qquad D_6 = \delta_1\beta_6 + \delta_{6u} + l\,\delta_5 + \delta_8,$$
$$D_7 = \delta_1\beta_7 + \delta_{7u} + m\,\delta_5, \qquad D_8 = \delta_1\beta_8 + \delta_{8u} + d\,\delta_5.$$

The system is now symmetrical under the substitutions (2) and (5) augmented by

$$(8) \qquad \begin{pmatrix} C \\ D \end{pmatrix}.$$

Computing the sixth order derivatives of x, we find after reduction that the expressions for $(x_{uuuuu})_v$ and $(x_{uuuuv})_u$ are identical, as are the expressions for $(x_{uuuuv})_v$

and $(x_{uuuvv})_u$. Hence by the symmetry the expressions for $(x_{uuvvv})_v$ and $(x_{uvvvv})_u$ are identical, and the expressions for $(x_{uvvvv})_v$ and $(x_{vvvvv})_u$ are identical. Differentiating (6) with respect to v and (7) with respect to u we find two values for x_{uuuvvv} which are formally distinct.

$$(x_{uuuvv})_v = A_0 x_{uuvv} + A_1 x_{uuv} + A_2 x_{uvv} + A_3 x_{uu} + A_4 x_{uv} + A_5 x_{vv} + A_6 x_u + A_7 x_v + A_8 x,$$

$$(x_{uuvvv})_u = B_0 x_{uuvv} + B_1 x_{uuv} + B_2 x_{uvv} + B_3 x_{uu} + B_4 x_{uv} + B_5 x_{vv} + B_6 x_u + B_7 x_v + B_8 x,$$

where

$$A_0 = a\ell' + a_v + C_1, \quad A_1 = aD_1 + C_{1v} + \delta_1 C_2 + C_3,$$

$$A_2 = aD_2 + C_{2v} + \delta_2 C_2 + C_4, \quad A_3 = aD_3 + C_{3v} + \delta_3 C_2 + a' C_5,$$

$$A_4 = aD_4 + C_{4v} + \delta_4 C_2 + \ell' C_5 + C_6,$$

$$A_5 = aD_5 + C_{5v} + \delta_5 C_2 + \ell' C_5 + C_7,$$

$$A_6 = aD_6 + C_{6v} + \delta_6 C_2 + \ell' C_5,$$

$$A_7 = aD_7 + C_{7v} + \delta_7 C_2 + m' C_5 + C_8,$$

$$A_8 = aD_8 + C_{8v} + \delta_8 C_2 + \alpha' C_5,$$

and where the B's are found from the A's by means of the substitutions (2), (5), and (8) augmented by

$$\begin{pmatrix} A \\ B \end{pmatrix},$$

the subscripts 0 remaining unchanged. Since these two values of x_{uuuvvv} must be the same we have

$$(9)\qquad \Gamma_0 x_{uuvv} + \Gamma_1 x_{uuv} + \Gamma_2 x_{uvv} + \Gamma_3 x_{uu} + \Gamma_4 x_{uv} + \Gamma_5 x_{vv} + \Gamma_6 x_u + \Gamma_7 x_v + \Gamma_8 x = 0,$$

where

$$\Gamma_0 = A_0 - B_0 = 3(a_v - b'_u),$$
$$\Gamma_i = A_i - B_i, \qquad (i = 1, \cdots, 8).$$

Two cases now arise as $a_v - b'_u$ is or is not equal to zero.

(a) $a_v - b'_u = 0$. Equation (9) now becomes a differential equation of the third order, and since by hypothesis x cannot be a solution of more than two independent equations of the third order, all the coefficients of (9) must vanish. This gives eight conditions on the coefficients of (1),

$$(10)\qquad \Gamma_i = 0, \qquad (i = 1, \cdots, 8),$$

which are the desired integrability conditions. In this case it is impossible to express x_{uuvv} in terms of x_{uuv}, x_{uvv}, and derivatives of lower orders including x. Since however all derivatives of higher orders are expressible linearly in terms of x_{uuvv}, x_{uuv}, x_{uvv}, and derivatives of lower orders, the integral surfaces of equations (1) will lie in a space S_8 determined by the nine independent points x_{uuvv}, x_{uuv}, x_{uvv}, x_{uu}, x_{uv}, x_{vv}, x_u, x_v, x.

(b) $a_v - b'_u \neq 0$. Equation (9) may now be solved for x_{uuvv} giving

$$(11)\qquad x_{uuvv} = \gamma_1 x_{uuv} + \gamma_2 x_{uvv} + \gamma_3 x_{uu} + \gamma_4 x_{uv} + \gamma_5 x_{vv} + \gamma_6 x_u + \gamma_7 x_v + \gamma_8 x,$$

where $\gamma_i = -\Gamma_i / \Gamma_0 \; ; \qquad (i = 1, \cdots, 8).$

In the substitution γ remains unchanged. Equations (6) and (7) may now be written

$$
\begin{aligned}
(12)\qquad x_{uuuvv} &= (C_1 + a\gamma_1)x_{uuv} + (C_2 + a\gamma_2)x_{uvv} \\
&\quad + (C_3 + a\gamma_3)x_{uu} + (C_4 + a\gamma_4)x_{uv} + (C_5 + a\gamma_5)x_{vv} \\
&\quad + (C_6 + a\gamma_6)x_u + (C_7 + a\gamma_7)x_v + (C_8 + a\gamma_8)x,
\end{aligned}
$$

$$
\begin{aligned}
(13)\qquad x_{uuvvv} &= (D_1 + b'\gamma_1)x_{uuv} + (D_2 + b'\gamma_2)x_{uvv} \\
&\quad + (D_3 + b'\gamma_3)x_{uu} + (D_4 + b'\gamma_4)x_{uv} + (D_5 + b'\gamma_5)x_{vv} \\
&\quad + (D_6 + b'\gamma_6)x_u + (D_7 + b'\gamma_7)x_v + (D_8 + b'\gamma_8)x.
\end{aligned}
$$

Differentiating equation (11) with respect to u and comparing the result with equation (12) we have two values for x_{uuuvv} which must be equal. Subtracting we have an equation of the third order whose coefficients must by hypothesis vanish. This gives the following eight integrability conditions:

$$
\begin{aligned}
C_1 + a\gamma_1 &= \gamma_1\beta_1 + \gamma_{1u} + \gamma_2\gamma_1 + \gamma_4, \\
C_2 + a\gamma_2 &= \gamma_1\beta_2 + \gamma_{2u} + \gamma_2^2 + \gamma_5, \\
C_3 + a\gamma_3 &= \gamma_1\beta_3 + \gamma_{3u} + \gamma_2\gamma_3 + a\gamma_3 + \gamma_6, \\
(14)\qquad C_4 + a\gamma_4 &= \gamma_1\beta_4 + \gamma_{4u} + \gamma_2\gamma_4 + l\gamma_3 + \gamma_7, \\
C_5 + a\gamma_5 &= \gamma_1\beta_5 + \gamma_{5u} + \gamma_2\gamma_5 + l\gamma_3, \\
C_6 + a\gamma_6 &= \gamma_1\beta_6 + \gamma_{6u} + \gamma_2\gamma_6 + l\gamma_3 + \gamma_8, \\
C_7 + a\gamma_7 &= \gamma_1\beta_7 + \gamma_{7u} + \gamma_2\gamma_7 + m\gamma_3, \\
C_8 + a\gamma_8 &= \gamma_1\beta_8 + \gamma_{8u} + \gamma_2\gamma_8 + d\gamma_3.
\end{aligned}
$$

Differentiating equation (11) with respect to v and comparing the result with equation (13) we get a second set of eight integrability conditions which may be found from equations (14) by applying the substitutions (2), (5), and (8). If these sixteen conditions are satisfied, then all derivatives of x of the fourth and higher orders may be expressed uniquely as linear combinations of x_{uuv}, x_{uvv}, x_{uu}, x_{uv}, x_{vv}, x_u, x_v, x, and the integral surfaces of equations (1) will lie in a space S_7 determined by these eight independent points.

2. The Second Canonical Form. The equations of the second canonical form are

$$\begin{aligned} x_{uuv} &= a x_{uu} + b x_{uv} + c x_{vv} + l x_u + m x_v + d x, \\ x_{vvv} &= a' x_{uu} + b' x_{uv} + c' x_{vv} + l' x_u + m' x_v + d' x. \end{aligned} \tag{15}$$

Differentiating each of these equations with respect to u and v we have

$$\begin{aligned} x_{uuuv} &= \beta_1 x_{uuu} + \beta_2 x_{uvv} + \beta_3 x_{uu} + \beta_4 x_{uv} + \beta_5 x_{vv} \\ &\quad + \beta_6 x_u + \beta_7 x_v + \beta_8 x, \\ x_{uuvv} &= \gamma_2 x_{uvv} + \gamma_3 x_{uu} + \gamma_4 x_{uv} + \gamma_5 x_{vv} \\ &\quad + \gamma_6 x_u + \gamma_7 x_v + \gamma_8 x, \\ x_{uvvv} &= \delta_1 x_{uuu} + \delta_2 x_{uvv} + \delta_3 x_{uu} + \delta_4 x_{uv} + \delta_5 x_{vv} \\ &\quad + \delta_6 x_u + \delta_7 x_v + \delta_8 x, \\ x_{vvvv} &= \epsilon_2 x_{uvv} + \epsilon_3 x_{uu} + \epsilon_4 x_{uv} + \epsilon_5 x_{vv} \\ &\quad + \epsilon_6 x_u + \epsilon_7 x_v + \epsilon_8 x, \end{aligned} \tag{16}$$

where

$$\beta_1 = a\,,\quad \beta_2 = b\,,\quad \beta_3 = ka + a_u + l,$$
$$\beta_4 = k^2 + k_u + m\,,\quad \beta_5 = kb + b_u\,,$$
$$\beta_6 = kl + l_u + d\,,\quad \beta_7 = km + m_u\,,$$
$$\beta_8 = kd + d_u\,,$$

$$\gamma_2 = k\,,\quad \gamma_3 = a^2 + a_v + ba'\,,\quad \gamma_4 = ak + k_v + bk' + l\,,$$
$$\gamma_5 = ab + b_v + bb' + m\,,\quad \gamma_6 = al + l_v + bl'\,,$$
$$\gamma_7 = am + m_v + bm' + d\,,\quad \gamma_8 = ad + d_v + bd'\,,$$

(17)
$$\delta_1 = a'\,,\quad \delta_2 = b'\,,\quad \delta_3 = k'a + a'_u + l'\,,$$
$$\delta_4 = k'k + k'_u + m'\,,\quad \delta_5 = k'b + b'_u\,,$$
$$\delta_6 = k'l + l'_u + d'\,,\quad \delta_7 = k'm + m'_u\,,$$
$$\delta_8 = k'd + d'_u\,,$$

$$\epsilon_2 = k'\,,\quad \epsilon_3 = a'a + a'_v + b'a'\,,\quad \epsilon_4 = a'k + k'_v + b'k' + l'\,,$$
$$\epsilon_5 = a'b + b'_v + b'^2 + m'\,,\quad \epsilon_6 = a'l + l'_v + b'l'\,,$$
$$\epsilon_7 = a'm + m'_v + b'm' + d'\,,\quad \epsilon_8 = a'd + d'_v + b'd'.$$

Computing the derivatives of x of fifth order as linear combinations of x_{uuuu}, x_{uuu}, x_{uvv}, and derivatives of lower orders, and expressing the coefficients in terms of the coefficients of (15) by means of (17), we find that the expressions for $(x_{uuuv})_v$ and $(x_{uuvv})_u$ are identical and that the expressions for $(x_{uvvv})_v$ and $(x_{vvvv})_u$ are identical. The two expressions for x_{uuvvv} are formally distinct.

(18)
$$(x_{uuvv})_v = A_1 x_{uuuu} + A_2 x_{uvv} + A_3 x_{uu} + A_4 x_{uv}$$

$$+ A_5 x_{vv} + A_6 x_u + A_7 x_v + A_8 x,$$

$$(x_{uvvv})_u = a' x_{uuuu} + B_1 x_{uuu} + B_2 x_{uvv} + B_3 x_{uu} + B_4 x_{uv}$$
$$+ B_5 x_{vv} + B_6 x_u + B_7 x_v + B_8 x,$$

where

$$A_1 = l a', \quad A_2 = l b' + l_v + y_4,$$
$$A_3 = l \delta_3 + y_{3v} + a y_3 + a' y_5,$$
$$A_4 = l \delta_4 + y_{4v} + l y_3 + l' y_5 + y_6,$$
$$A_5 = l \delta_5 + y_{5v} + b y_3 + b' y_5 + y_7,$$
$$A_6 = l \delta_6 + y_{6v} + l y_3 + l' y_5,$$
$$A_7 = l \delta_7 + y_{7v} + m y_3 + m' y_5 + y_8,$$
$$A_8 = l \delta_8 + y_{8v} + d y_3 + d' y_5,$$

$$B_1 = a'_u + \delta_3, \quad B_2 = b' l + l'_u + \delta_5,$$
$$B_3 = b' y_3 + \delta_{3u} + a \delta_4 + \delta_6, \quad B_4 = b' y_4 + \delta_{4u} + l \delta_4 + \delta_7,$$
$$B_5 = b' y_5 + \delta_{5u} + b \delta_4, \quad B_6 = b' y_6 + \delta_{6u} + l \delta_4 + \delta_8,$$
$$B_7 = b' y_7 + \delta_{7u} + m \delta_4, \quad B_8 = b' y_8 + \delta_{8u} + d \delta_4.$$

Two cases now arise as $a' = 0$ or $a' \neq 0$.

(a) $a' = 0$. Equating the two values of x_{uuvvv} given by equations (18) we have a third equation of the third order in x independent of equations (15). Hence the coefficients of this equation must vanish giving the set of eight integrability conditions,

(19) $$A_i - B_i = 0, \quad (i = 1, \cdots, 8).$$

In this case it is impossible to express x_{uuuu} as a linear com-

bination of x_{uuu}, x_{uvv}, and derivatives of lower orders, and therefore impossible to express any higher derivatives of x with respect to u alone as such a combination. The integral surfaces of equations (15) lie in a general space S_n with $n \geq 7$.

(b) $a' \neq 0$. Solving equations (18) for x_{uuuu} we find

$$x_{uuuu} = \alpha_1 x_{uuu} + \alpha_2 x_{uvv} + \alpha_3 x_{uu} + \alpha_4 x_{uv} + \alpha_5 x_{vv} + \alpha_6 x_u + \alpha_7 x_v + \alpha_8 x, \tag{20}$$

where $\alpha_i = (A_i - B_i)/a'$, $(i = 1, \cdots, 8)$.

The integrability conditions are found by equating the value of $(x_{uuuv})_u$ obtained by differentiating the first equation of (16) with respect to u and the value of $(x_{uuuu})_v$ obtained by differentiating equation (20) with respect to v. This gives the following set of eight conditions:

$$\begin{aligned}
&a_u + \beta_3 = \alpha_{1v} + a'\alpha_2,\\
&a\alpha_2 + b_u + bh + \beta_5 = b\alpha_1 + \alpha_{2v} + b'\alpha_2 + \alpha_4,\\
&\beta_{3u} + b\gamma_3 + a\beta_4 + \beta_6 = \alpha_1\beta_3 + \alpha_{3v} + \alpha_2\delta_3 + a'\alpha_5,\\
&a\alpha_4 + \beta_{4u} + b\gamma_4 + h\beta_4 + \beta_7 = \alpha_1\beta_4 + \alpha_{4v} + \alpha_2\delta_4 + h\alpha_3 + h'\alpha_5 + \alpha_6,\\
&a\alpha_5 + \beta_{5u} + b\gamma_5 + h\beta_4 = \alpha_1\beta_5 + \alpha_{5v} + \alpha_2\delta_5 + b\alpha_3 + b'\alpha_5 + \alpha_7,\\
&a\alpha_6 + \beta_{6u} + b\gamma_6 + h\beta_4 + \beta_8 = \alpha_1\beta_6 + \alpha_{6v} + \alpha_2\delta_6 + h\alpha_3 + h'\alpha_5,
\end{aligned} \tag{21}$$

$$
\begin{aligned}
a\alpha_7 + \beta_{7u} + b\gamma_7 + m\beta_4 &\\
&= \alpha_1\beta_7 + \alpha_{7v} + \alpha_2\delta_7 + m\alpha_3 + m'\alpha_5 + \alpha_8,\\
a\alpha_8 + \beta_{8u} + b\gamma_8 + d\beta_4 &\\
&= \alpha_1\beta_8 + \alpha_{8v} + \alpha_2\delta_8 + d\alpha_3 + d'\alpha_5.
\end{aligned}
$$

If these conditions are satisfied, all derivatives of x of the fourth and higher orders can be expressed uniquely as linear combinations of x_{uuu}, x_{uvv}, x_{uu}, x_{uv}, x_{vv}, x_u, x_v, x, and the integral surfaces of equations (15) will be immersed in the space S_7 determined by these eight points.

3. The Third Canonical Form. The equations of the third canonical form are

$$
\begin{aligned}
x_{uuu} &= a x_{uu} + h x_{uv} + b x_{vv} + l x_u + m x_v + d x,\\
x_{uuv} &= a' x_{uu} + h' x_{uv} + b' x_{vv} + l' x_u + m' x_v + d' x.
\end{aligned}
\tag{22}
$$

The integrability conditions are found by differentiating the first of these with respect to v and the second with respect to u and equating the two values of x_{uuuv} thus obtained. These conditions are

$$
\begin{aligned}
h &= b',\\
b &= 0,\\
a_v &= a'_u + a'h' + l',\\
h_v + ah' + l &= h'_u + a'h + h'^2 + m',\\
ab' + m &= b'_u + h'b',\\
l_v + al' &= l'_u + a'l + h'l' + d',\\
m_v + am' + d &= m'_u + a'm + h'm',\\
d_v + ad' &= d'_u + a'd + h'd'.
\end{aligned}
\tag{23}
$$

This set of conditions has already been computed by Lane.[1] Since it is impossible to express x_{uvvv} and x_{vvvv} as linear combinations of derivatives of x of lower orders, and therefore impossible to express all higher derivatives of x obtained from these by differentiation with respect to v alone as such combinations, the integral surfaces of equations (22) will be immersed in a space S_n with $n \geq 7$.

With $b = 0$, x_{uuuu} may be expressed as a linear combination of x_{uu}, x_{uv}, x_{vv}, x_u, x_v, x. This fact, together with the first equation of (22), gives the theorem: The osculating space S_4 at each point of a u curve on an integral surface S of equations (22) lies in the osculating space S(2,0) of S at the point.

4. The Fourth Canonical Form. The equations of the fourth canonical form are

$$\begin{aligned} x_{uuv} &= a x_{uu} + k x_{uv} + b x_{vv} + l x_u + m x_v + d x, \\ x_{uvv} &= a' x_{uu} + k' x_{uv} + b' x_{vv} + l' x_u + m' x_v + d' x. \end{aligned} \tag{24}$$

The integrability conditions are found by differentiating the first of these with respect to v and the second with respect to u and equating the two values of x_{uuvv} thus obtained. These conditions are

$$\begin{aligned} a' &= 0, \\ b &= 0, \\ a_v + a^2 &= k' a + l', \end{aligned}$$

---

[1] E. P. Lane, loc. cit.

$$
\begin{aligned}
(25)\qquad h_v + ah + l &= h'_u + b'h' + m', \\
h b' + m &= b'_u + b'^2, \\
l_v + al + hl' &= l'_u + h'l + b'l' + d', \\
m_v + am + hm' + d &= m'_u + h'm + b'm', \\
d_v + ad + hd' &= d'_u + h'd + b'd'.
\end{aligned}
$$

These conditions have already been calculated by Bompiani.[2] Since it is impossible to express x_{uuuu} and x_{vvvv} as linear combinations of derivatives of x of lower orders, and so impossible to express any higher derivative of x with respect to u alone or with respect to v alone as such a combination, the integral surfaces of equations (24) will lie in a space S_n with $n \geq 7$.

5. The Fifth Canonical Form. The equations of the fifth canonical form are

$$
\begin{aligned}
x_{uuv} &= a x_{uu} + h x_{uv} + b x_{vv} + l x_u + m x_v + d x, \\
(26)\qquad x_{uvv} &= D x_{vvv} + a' x_{uu} + h' x_{uv} + b' x_{vv} + l' x_u \\
&\quad + m' x_v + d' x, \qquad D \neq 0.
\end{aligned}
$$

Differentiating each of these equations with respect to u and v, we get four equations involving all of the derivatives of x of the fourth order except x_{uuuu}. These equations may be solved for these four fourth order derivatives, giving

---

[2] E. Bompiani, Determinazione della superficie integrali d'un sistema di equazione a derivate parziali lineari ed omogenee, Rend. del R. Istituto Lombardo di Scienze e Lettere, vol. 52 (1919), p. 826.

$$
\begin{aligned}
x_{uuuv} &= \beta_1 x_{uuu} + \beta_2 x_{vvv} + \beta_3 x_{uu} + \beta_4 x_{uv} + \beta_5 x_{vv} \\
&\quad + \beta_6 x_u + \beta_7 x_v + \beta_8 x, \\
x_{uuvv} &= \gamma_2 x_{vvv} + \gamma_3 x_{uu} + \gamma_4 x_{uv} + \gamma_5 x_{vv} \\
&\quad + \gamma_6 x_u + \gamma_7 x_v + \gamma_8 x, \\
x_{uvvv} &= \delta_1 x_{uuu} + \delta_2 x_{vvv} + \delta_3 x_{uu} + \delta_4 x_{uv} + \delta_5 x_{vv} \\
&\quad + \delta_6 x_u + \delta_7 x_v + \delta_8 x, \\
x_{vvvv} &= \epsilon_1 x_{uuu} + \epsilon_2 x_{vvv} + \epsilon_3 x_{uu} + \epsilon_4 x_{uv} + \epsilon_5 x_{vv} \\
&\quad + \epsilon_6 x_u + \epsilon_7 x_v + \epsilon_8 x,
\end{aligned}
\tag{27}
$$

where

$$
\begin{aligned}
&\beta_1 = a, \quad \beta_2 = Db, \quad \beta_3 = a_u + ha + ba' + l, \\
&\beta_4 = h_u + h^2 + bh' + m, \quad \beta_5 = b_u + hb + bb', \\
&\beta_6 = l_u + hl + bl' + d, \quad \beta_7 = m_u + hm + bm', \\
&\beta_8 = d_u + hd + bd',
\end{aligned}
$$

$$
\begin{aligned}
&\gamma_2 = hD + b, \quad \gamma_3 = a_v + a^2 + ha', \\
&\gamma_4 = h_v + ah + hh' + l, \quad \gamma_5 = b_v + ab + hb' + m, \\
&\gamma_6 = l_v + al + hl', \quad \gamma_7 = m_v + am + hm' + d, \\
&\gamma_8 = d_v + ad + hd',
\end{aligned}
$$

$$
\begin{aligned}
&\delta_1 = -a'/D, \quad \delta_2 = \tfrac{1}{D}(\gamma_2 - D_u - Db'), \\
&\delta_3 = \tfrac{1}{D}(\gamma_3 - a'_u - a'b' - h'a - l'), \\
&\delta_4 = \tfrac{1}{D}(\gamma_4 - h'_u - b'h' - h'h - m'), \\
&\delta_5 = \tfrac{1}{D}(\gamma_5 - b'_u - b'^2 - h'b), \\
&\delta_6 = \tfrac{1}{D}(\gamma_6 - l'_u - b'l' - h'l - d'), \\
&\delta_7 = \tfrac{1}{D}(\gamma_7 - m'_u - b'm' - h'm), \\
&\delta_8 = \tfrac{1}{D}(\gamma_8 - d'_u - b'd' - h'd),
\end{aligned}
\tag{28}
$$

$$\epsilon_1 = \frac{1}{D}\,\delta_1 \ ,$$

$$\epsilon_2 = \frac{1}{D}\,(\delta_2 - D_v - \ell' D - b'),$$

$$\epsilon_3 = \frac{1}{D}\,(\delta_3 - a'_v - a'a - \ell' a'),$$

$$\epsilon_4 = \frac{1}{D}\,(\delta_4 - \ell'_v - a'\ell - \ell'^2 - \ell'),$$

$$\epsilon_5 = \frac{1}{D}\,(\delta_5 - b'_v - a'b - \ell' b' - m'),$$

$$\epsilon_6 = \frac{1}{D}\,(\delta_6 - \ell'_v - a'\ell - \ell'\ell'),$$

$$\epsilon_7 = \frac{1}{D}\,(\delta_7 - m'_v - a'm - \ell' m' - d'),$$

$$\epsilon_8 = \frac{1}{D}\,(\delta_8 - d'_v - a'd - \ell' d').$$

Computing the values of $(x_{uuvv})_v$, $(x_{uvvv})_u$, $(x_{uvvv})_v$, and $(x_{vvvv})_u$ we have

(29) $$(x_{uuvv})_v = A_1 x_{uuu} + A_2 x_{vvv} + A_3 x_{uu} + A_4 x_{uv} + A_5 x_{vv} + A_6 x_u + A_7 x_v + A_8 x,$$

(30) $$(x_{uvvv})_u = (-a'/D)\,x_{uuuu} + B_1 x_{uuu} + B_2 x_{vvv} + B_3 x_{uu} + B_4 x_{uv} + B_5 x_{vv} + B_6 x_u + B_7 x_v + B_8 x,$$

(31) $$(x_{uvvv})_v = C_1 x_{uuu} + C_2 x_{vvv} + C_3 x_{uu} + C_4 x_{uv} + C_5 x_{vv} + C_6 x_u + C_7 x_v + C_8 x,$$

(32) $$(x_{vvvv})_u = (-a'/D^2)\,x_{uuuu} + D_1 x_{uuu} + D_2 x_{vvv} + D_3 x_{uu} + D_4 x_{uv} + D_5 x_{vv} + D_6 x_u + D_7 x_v + D_8 x,$$

where

$$A_1 = \gamma_2 \epsilon_1 , \quad A_2 = \gamma_{2v} + \gamma_2 \epsilon_2 + D\gamma_4 + \gamma_5 ,$$
$$A_3 = \gamma_{3v} + \gamma_2 \epsilon_3 + a\gamma_3 + a'\gamma_4 ,$$
$$A_4 = \gamma_{4v} + \gamma_2 \epsilon_4 + h\gamma_3 + h'\gamma_4 + \gamma_6 ,$$
$$A_5 = \gamma_{5v} + \gamma_2 \epsilon_5 + b\gamma_3 + b'\gamma_4 + \gamma_7 ,$$
$$A_6 = \gamma_{6v} + \gamma_2 \epsilon_6 + l\gamma_3 + l'\gamma_4 ,$$
$$A_7 = \gamma_{7v} + \gamma_2 \epsilon_7 + m\gamma_3 + m'\gamma_4 + \gamma_8 ,$$
$$A_8 = \gamma_{8v} + \gamma_2 \epsilon_8 + d\gamma_3 + d'\gamma_4 ,$$

$$B_1 = \delta_{1u} + \delta_1 \delta_2 + \delta_3 , \quad B_2 = \delta_{2u} + \delta_2^2 + D\delta_5 ,$$
$$B_3 = \delta_{3u} + \delta_2 \delta_3 + a\delta_4 + a'\delta_5 + \delta_6 ,$$
$$B_4 = \delta_{4u} + \delta_2 \delta_4 + h\delta_4 + h'\delta_5 + \delta_7 ,$$
$$B_5 = \delta_{5u} + \delta_2 \delta_5 + b\delta_4 + b'\delta_5 ,$$
$$B_6 = \delta_{6u} + \delta_2 \delta_6 + l\delta_4 + l'\delta_5 + \delta_8 ,$$
$$B_7 = \delta_{7u} + \delta_2 \delta_7 + m\delta_4 + m'\delta_5 ,$$
$$B_8 = \delta_{8u} + \delta_2 \delta_8 + d\delta_4 + d'\delta_5 ,$$

(33)

$$C_1 = \delta_{1v} + \delta_1 \beta_1 + \delta_2 \epsilon_1 ,$$
$$C_2 = \delta_{2v} + \delta_1 \beta_2 + \delta_2 \epsilon_2 + D\delta_4 + \delta_5 ,$$
$$C_3 = \delta_{3v} + \delta_1 \beta_3 + \delta_2 \epsilon_3 + a\delta_3 + a'\delta_4 ,$$
$$C_4 = \delta_{4v} + \delta_1 \beta_4 + \delta_2 \epsilon_4 + h\delta_3 + h'\delta_4 + \delta_6 ,$$
$$C_5 = \delta_{5v} + \delta_1 \beta_5 + \delta_2 \epsilon_5 + b\delta_3 + b'\delta_4 + \delta_7 ,$$
$$C_6 = \delta_{6v} + \delta_1 \beta_6 + \delta_2 \epsilon_6 + l\delta_3 + l'\delta_4 ,$$
$$C_7 = \delta_{7v} + \delta_1 \beta_7 + \delta_2 \epsilon_7 + m\delta_3 + m'\delta_4 + \delta_8 ,$$
$$C_8 = \delta_{8v} + \delta_1 \beta_8 + \delta_2 \epsilon_8 + d\delta_3 + d'\delta_4 ,$$

$$
\begin{aligned}
\mathcal{D}_1 &= \epsilon_{1u} + \epsilon_2\delta_1 + \epsilon_3 ,\\
\mathcal{D}_2 &= \epsilon_{2u} + \epsilon_2\delta_2 + \mathcal{D}\epsilon_5 ,\\
\mathcal{D}_3 &= \epsilon_{3u} + \epsilon_2\delta_3 + a\epsilon_4 + a'\epsilon_5 + \epsilon_6 ,\\
\mathcal{D}_4 &= \epsilon_{4u} + \epsilon_2\delta_4 + h\epsilon_4 + h'\epsilon_5 + \epsilon_7 ,\\
\mathcal{D}_5 &= \epsilon_{5u} + \epsilon_2\delta_5 + b\epsilon_4 + b'\epsilon_5 ,\\
\mathcal{D}_6 &= \epsilon_{6u} + \epsilon_2\delta_6 + \ell\epsilon_4 + \ell'\epsilon_5 + \epsilon_8 ,\\
\mathcal{D}_7 &= \epsilon_{7u} + \epsilon_2\delta_7 + m\epsilon_4 + m'\epsilon_5 ,\\
\mathcal{D}_8 &= \epsilon_{8u} + \epsilon_2\delta_8 + d\epsilon_4 + d'\epsilon_5 .
\end{aligned}
$$

Two cases now arise as $a' \neq 0$ or $a' = 0$.

(a) $a' \neq 0$. Equations (29) and (30) can be solved for x_{uuuu} as a linear combination of x_{uuu}, x_{vvv}, and derivatives of x of lower orders. Similarly equations (31) and (32) can be solved for x_{uuuu}. On reducing the coefficients of these two expressions by means of (33) and (28) we find that the two expressions for x_{uuuu} are identical. Hence we have a unique value for x_{uuuu} which we write

$$
\begin{aligned}
(34)\qquad x_{uuuu} = {} & \alpha_1 x_{uuu} + \alpha_2 x_{vvv} + \alpha_3 x_{uu} + \alpha_4 x_{uv} + \alpha_5 x_{vv}\\
& + \alpha_6 x_u + \alpha_7 x_v + \alpha_8 x ,
\end{aligned}
$$

where

$$
(35)\qquad \alpha_i = (\mathcal{D}/a')(B_i - A_i) = (\mathcal{D}^2/a')(\mathcal{D}_i - C_i) ,
$$
$$
(i = 1, \cdots, 8).
$$

In computing the integrability conditions we find that the two expressions for x_{uuuvv} found by differentiating the first two equations of (27) with respect to v and u respectively are identical. The two expressions for x_{uuuuv} found by differentiating

the first equation of (27) with respect to u and equation (34) with respect to v are not identical, and give rise to the following eight integrability conditions:

$$
\begin{aligned}
&\beta_{1u} + \beta_2\delta_1 + \beta_3 = \alpha_{1v} + \alpha_2\epsilon_1,\\
&\beta_{2u} + \beta_1\alpha_2 + \beta_2\delta_2 + D\beta_5\\
&\qquad = \alpha_{2v} + \alpha_1\beta_2 + \alpha_2\epsilon_2 + D\alpha_4 + \alpha_5,\\
&\beta_{3u} + \beta_1\alpha_3 + \beta_2\delta_3 + a\beta_4 + a'\beta_5 + \beta_6\\
&\qquad = \alpha_{3v} + \alpha_1\beta_3 + \alpha_2\epsilon_3 + a\alpha_3 + a'\alpha_4,\\
&\beta_{4u} + \beta_1\alpha_4 + \beta_2\delta_4 + k\beta_4 + k'\beta_5 + \beta_7\\
&\qquad = \alpha_{4v} + \alpha_1\beta_4 + \alpha_2\epsilon_4 + k\alpha_3 + k'\alpha_4 + \alpha_6,\\
&\beta_{5u} + \beta_1\alpha_5 + \beta_2\delta_5 + b\beta_4 + b'\beta_5\\
&\qquad = \alpha_{5v} + \alpha_1\beta_5 + \alpha_2\epsilon_5 + b\alpha_3 + b'\alpha_4 + \alpha_7,\\
&\beta_{6u} + \beta_1\alpha_6 + \beta_2\delta_6 + l\beta_4 + l'\beta_5 + \beta_8\\
&\qquad = \alpha_{6v} + \alpha_1\beta_6 + \alpha_2\epsilon_6 + l\alpha_3 + l'\alpha_4,\\
&\beta_{7u} + \beta_1\alpha_7 + \beta_2\delta_7 + m\beta_4 + m'\beta_5\\
&\qquad = \alpha_{7v} + \alpha_1\beta_7 + \alpha_2\epsilon_7 + m\alpha_3 + m'\alpha_4 + \alpha_8,\\
&\beta_{8u} + \beta_1\alpha_8 + \beta_2\delta_8 + d\beta_4 + d'\beta_5\\
&\qquad = \alpha_{8v} + \alpha_1\beta_8 + \alpha_2\epsilon_8 + d\alpha_3 + d'\alpha_4.
\end{aligned}
\tag{36}
$$

In this case all derivatives of x of fourth and higher orders can be expressed as linear combinations of x_{uuu}, x_{vvv}, x_{uu}, x_{uv}, x_{vv}, x_u, x_v, x. The integral surfaces of equations (26) are therefore immersed in a space S_7 determined by these eight independent points.

The first of the integrability conditions reduces to

$$(37) \qquad (\alpha_1 + \delta_2 + \mathcal{L} + b')_v = (\beta_1 + \epsilon_2 + a + \mathcal{L}')_u.$$

This relation can also be found in the following way. Since x_{uuu}, x_{vvv}, x_{uu}, x_{uv}, x_{vv}, x_u, x_v, x are independent points in the space S_7 which is the ambient space of the integral surfaces of equations (26), the determinant of their coefficients does not vanish. Denote this determinant by W. Then if k is a constant,

$$(\log k W)_u = \alpha_1 + \delta_2 + \mathcal{L} + b',$$
$$(\log k W)_v = \beta_1 + \epsilon_2 + a + \mathcal{L}',$$

whence the relation (37) follows immediately. Thus there exists a function,

$$(38) \qquad p = \log k W,$$

such that

$$p_u = \alpha_1 + \delta_2 + \mathcal{L} + b',$$
$$p_v = \beta_1 + \epsilon_2 + a + \mathcal{L}'.$$

(b) $a' = 0$. In this case it is impossible to solve equations (29) to (32) for x_{uuuu}. The integrability conditions are found by equating the two values of x_{uuvvv} given by equations (29) and (30), and by equating the two values of x_{uvvvv} given by equations (31) and (32). But since from (35) we have

$$(B_i - A_i) = D\,(D_i - C_i), \quad (i = 1, \cdots, 8),$$

there are just eight integrability conditions which may be

written in either of the two forms:

(39) $B_i - A_i = 0, \qquad (i = 1,---,8),$

or

(40) $D_i - C_i = 0. \qquad (i = 1,---,8).$

In this case it is impossible to express x_{uuuu}, and therefore any derivative of x of higher order with respect to u alone, as a linear combination of derivatives of lower orders, and hence the integral surfaces of equations (26) will be immersed in a general space S_n with $n \geq 7$.

6. The Sixth Canonical Form. The differential equations of the sixth canonical form are

$$(41)\qquad \begin{aligned} x_{uuv} &= a\,x_{uu} + b\,x_{uv} + c\,x_{vv} + l\,x_u + m\,x_v + d\,x, \\ x_{vvv} &= 4C^3 x_{uuu} - 3C\,x_{uvv} + a'x_{uu} + b'x_{uv} + c'x_{vv} \\ &\quad + l'x_u + m'x_v + d'x, \qquad C \neq 0. \end{aligned}$$

By differentiation and substitution we get from these the following expressions for x_{uuuv}, x_{uuvv}, x_{uvvv}, x_{vvvv}:

$$(42)\qquad \begin{aligned} x_{uuuv} &= \beta_1 x_{uuu} + \beta_2 x_{uvv} + \beta_3 x_{uu} + \beta_4 x_{uv} + \beta_5 x_{vv} \\ &\quad + \beta_6 x_u + \beta_7 x_v + \beta_8 x, \\ x_{uuvv} &= \gamma_1 x_{uuu} + \gamma_2 x_{uvv} + \gamma_3 x_{uu} + \gamma_4 x_{uv} + \gamma_5 x_{vv} \\ &\quad + \gamma_6 x_u + \gamma_7 x_v + \gamma_8 x, \\ x_{uvvv} &= 4C^3 x_{uuuu} + \delta_1 x_{uuu} + \delta_2 x_{uvv} + \delta_3 x_{uu} + \delta_4 x_{uv} \\ &\quad + \delta_5 x_{vv} + \delta_6 x_u + \delta_7 x_v + \delta_8 x, \end{aligned}$$

$$x_{vvvv} = -12C^4 x_{uuuuu} + \epsilon_1 x_{uuuu} + \epsilon_2 x_{uvv} + \epsilon_3 x_{uu} + \epsilon_4 x_{uv} + \epsilon_5 x_{vv} + \epsilon_6 x_u + \epsilon_7 x_v + \epsilon_8 x,$$

where

$$\beta_1 = a, \quad \beta_2 = b, \quad \beta_3 = a_u + ah + l,$$
$$\beta_4 = h_u + h^2 + m, \quad \beta_5 = b_u + hb,$$
$$\beta_6 = l_u + hl + d, \quad \beta_7 = m_u + hm,$$
$$\beta_8 = d_u + hd,$$

$$\gamma_1 = 4bC^3, \quad \gamma_2 = h - 3bC,$$
$$\gamma_3 = a_v + a^2 + a'b, \quad \gamma_4 = h_v + ah + bh' + l,$$
$$\gamma_5 = b_v + ab + bb' + m, \quad \gamma_6 = l_v + al + bl',$$
$$\gamma_7 = m_v + am + bm' + d, \quad \gamma_8 = d_v + ad + bd',$$

$$\delta_1 = 12C^2 C_u + a' - 3C\gamma_1,$$
$$\delta_2 = -3C_u + b' - 3C\gamma_2,$$
$$\delta_3 = a'_u + ah' + l' - 3C\gamma_3,$$
$$\delta_4 = h'_u + hh' + m' - 3C\gamma_4,$$
(43)
$$\delta_5 = b'_u + bh' - 3C\gamma_5,$$
$$\delta_6 = l'_u + lh' + d' - 3C\gamma_6,$$
$$\delta_7 = m'_u + mh' - 3C\gamma_7,$$
$$\delta_8 = d'_u + dh' - 3C\gamma_8,$$

$$\epsilon_1 = 12C^2 C_v + 4b'C^3 + 4C^3\beta_1 - 3C\delta_1,$$
$$\epsilon_2 = -3C_v + h' - 3b'C + 4C^3\beta_2 - 3C\delta_2,$$
$$\epsilon_3 = a'_v + aa' + a'b' + 4C^3\beta_3 - 3C\delta_3,$$
$$\epsilon_4 = h'_v + a'h + b'h' + l' + 4C^3\beta_4 - 3C\delta_4,$$

$$\epsilon_5 = b'_v + a'b + b'^2 + m' + 4C^3\beta_5 - 3C\delta_5,$$
$$\epsilon_6 = l'_v + a'l + b'l' + 4C^3\beta_6 - 3C\delta_6,$$
$$\epsilon_7 = m'_v + a'm + b'm' + d' + 4C^3\beta_7 - 3C\delta_7,$$
$$\epsilon_8 = d'_v + a'd + b'd' + 4C^3\beta_8 - 3C\delta_8.$$

Differentiating each of these with respect to u and v we find unique values for all the derivatives of x of fifth order as linear combinations of x_{uuuu}, x_{uuu}, x_{uvv}, and derivatives of x of lower orders. The expressions for x_{uuuvv}, x_{uuvvv}, and x_{uvvvv} may be written in two forms which are found to be identical when their coefficients are expressed by means of (43) in terms of the coefficients of equations (41). The only ones of these which we shall need in the later computation are

(44)
$$x_{uuuuu} = A_0 x_{uuuu} + A_1 x_{uuu} + A_2 x_{uvv} + A_3 x_{uu} + A_4 x_{uv} + A_5 x_{vv} + A_6 x_u + A_7 x_v + A_8 x,$$
$$x_{uuuuv} = B_0 x_{uuuu} + B_1 x_{uuu} + B_2 x_{uvv} + B_3 x_{uu} + B_4 x_{uv} + B_5 x_{vv} + B_6 x_u + B_7 x_v + B_8 x,$$

where

$$A_0 = \frac{1}{4C^3}(4C^3\gamma_2 - 12C^2C_u - \delta_1),$$
$$A_1 = \frac{1}{4C^3}(\gamma_{1v} + \gamma_1\beta_1 + \gamma_2\delta_1 + 4C^3\delta_5 - \delta_{1u} - \delta_2\gamma_1 - \delta_3),$$
$$A_2 = \frac{1}{4C^3}(\gamma_{2v} + \gamma_1\beta_2 + \gamma_4 - 3C\gamma_5 - \delta_{2u} - \delta_5),$$
$$A_3 = \frac{1}{4C^3}(\gamma_{3v} + \gamma_1\beta_3 + \gamma_2\delta_3 + a\gamma_3 + a'\gamma_5 - \delta_{3u} - \delta_2\gamma_3 - a\delta_4 - \delta_6),$$

$$A_4 = \frac{1}{4C^3}\,(\gamma_{4v} + \gamma_1\beta_4 + \gamma_2\delta_4 + l\gamma_3 + l'\gamma_5 + \gamma_6 - \delta_{4u} - \delta_2\gamma_4 - l\delta_4 - \delta_7),$$

$$A_5 = \frac{1}{4C^3}\,(\gamma_{5v} + \gamma_1\beta_5 + \gamma_2\delta_5 + l\gamma_3 + l'\gamma_5 + \gamma_7 - \delta_{5u} - \delta_2\gamma_5 - l\delta_4),$$

$$A_6 = \frac{1}{4C^3}\,(\gamma_{6v} + \gamma_1\beta_6 + \gamma_2\delta_6 + l\gamma_3 + l'\gamma_5 - \delta_{6u} - \delta_2\gamma_6 - l\delta_4 - \delta_8),$$

$$A_7 = \frac{1}{4C^3}\,(\gamma_{7v} + \gamma_1\beta_7 + \gamma_2\delta_7 + m\gamma_3 + m'\gamma_5 + \gamma_8 - \delta_{7u} - \delta_2\gamma_7 - m\delta_4),$$

(45)

$$A_8 = \frac{1}{4C^3}\,(\gamma_{8v} + \gamma_1\beta_8 + \gamma_2\delta_8 + a\gamma_3 + a'\gamma_5 - \delta_{8u} - \delta_2\gamma_8 - a\delta_4),$$

$$B_0 = \beta_1,$$

$$B_1 = \beta_{1u} + \beta_2\gamma_1 + \beta_3,$$

$$B_2 = \beta_{2u} + \beta_2\gamma_2 + \beta_5,$$

$$B_3 = \beta_{3u} + \beta_2\gamma_3 + a\beta_4 + \beta_6,$$

$$B_4 = \beta_{4u} + \beta_2\gamma_4 + l\beta_4 + \beta_7,$$

$$B_5 = \beta_{5u} + \beta_2\gamma_5 + l\beta_4,$$

$$B_6 = \beta_{6u} + \beta_2\gamma_6 + l\beta_4 + \beta_8,$$

$$B_7 = \beta_{7u} + \beta_2\gamma_7 + m\beta_4,$$

$$B_8 = \beta_{8u} + \beta_2\gamma_8 + a\beta_4.$$

Differentiating these fifth order derivatives of x with respect to u and v we can express each derivative of the sixth order as a linear combination of x_{uuuu}, x_{uuu}, x_{uvv}, and derivatives of lower orders. The values of $(x_{uuuuv})_v$ and $(x_{uuuvv})_u$ are found to be identical, as are the values of $(x_{uuuvv})_v$ and $(x_{uuvvv})_u$, when the coefficients are expressed by means of (43) in terms of the coefficients of equations (41). The pairs of values of the other derivatives that can be computed in two ways are not identical, but it is found on reducing the coefficients that the three differences, $(x_{uuuuu})_v - (x_{uuuuv})_u$, $(x_{uuvvv})_v - (x_{uvvvv})_u$, and $(x_{uvvvv})_v - (x_{vvvvv})_u$, when expressed as linear combinations of x_{uuuu}, x_{uuu}, x_{uvv}, and derivatives of lower orders, are proportional. More precisely we have:

$$(46)\qquad (x_{uuuuu})_v - (x_{uuuuv})_u$$
$$= (1/4C^3)\left[(x_{uuvvv})_v - (x_{uvvvv})_u\right]$$
$$= -(1/12C^4)\left[(x_{uvvvv})_v - (x_{vvvvv})_u\right].$$

Hence the integrability conditions can be found by considering only the two expressions for x_{uuuuuv}. Forming the difference $(x_{uuuuu})_v - (x_{uuuuv})_u$, which must be zero, we have

$$(47)\qquad K_0 x_{uuuu} + K_1 x_{uuu} + K_2 x_{uvv} + K_3 x_{uu} + K_4 x_{uv} + K_5 x_{vv} + K_6 x_u + K_7 x_v + K_8 x = 0,$$

where

$$K_0 = A_{0v} + 4C^3 A_2 - B_{0u} - B_1,$$
$$K_1 = A_{1v} + A_0 B_1 + \beta_1 A_1 + \delta_1 A_2 + 4C^3 A_5 - B_{1u} - B_0 A_1 - \gamma_1 B_2 - B_3,$$

$$
\begin{aligned}
K_2 &= A_{2v} + A_0 B_2 + \beta_2 A_1 + \delta_2 A_2 + A_4 - 3C A_5 \\
&\quad - B_{2u} - B_0 A_2 - \gamma_2 B_2 - B_5, \\
K_3 &= A_{3v} + A_0 B_3 + \beta_3 A_1 + \delta_3 A_2 + a A_3 + a' A_5 \\
&\quad - B_{3u} - B_0 A_3 - \gamma_3 B_2 - a B_4 - B_6, \\
K_4 &= A_{4v} + A_0 B_4 + \beta_4 A_1 + \delta_4 A_2 + \ell A_3 + \ell' A_5 + A_6 \\
&\quad - B_{4u} - B_0 A_4 - \gamma_4 B_2 - \ell B_4 - B_7, \\
K_5 &= A_{5v} + A_0 B_5 + \beta_5 A_1 + \delta_5 A_2 + \ell A_3 + \ell' A_5 + A_7 \\
&\quad - B_{5u} - B_0 A_5 - \gamma_5 B_2 - \ell B_4, \\
K_6 &= A_{6v} + A_0 B_6 + \beta_6 A_1 + \delta_6 A_2 + \ell A_3 + \ell' A_5 \\
&\quad - B_{6u} - B_0 A_6 - \gamma_6 B_2 - \ell B_4 - B_8, \\
K_7 &= A_{7v} + A_0 B_7 + \beta_7 A_1 + \delta_7 A_2 + m A_3 + m' A_5 + A_8 \\
&\quad - B_{7u} - B_0 A_7 - \gamma_7 B_2 - m B_4, \\
K_8 &= A_{8v} + A_0 B_8 + \beta_8 A_1 + \delta_8 A_2 + d A_3 + d' A_5 \\
&\quad - B_{8u} - B_0 A_8 - \gamma_8 B_2 - d B_4.
\end{aligned}
\tag{48}
$$

K_0 may also be written:

$$
\begin{aligned}
K_0 &= (3\gamma_2 - 2\delta_1/4C^3 + a'/4C^3)_v - (\delta_2 + \ell' + 3a)_u \\
&= (A_0 + \gamma_2 + \ell)_v - (B_0 + \beta_1 + \delta_2 + a + \ell')_u.
\end{aligned}
$$

Two cases now arise as $K_0 = 0$ or $K_0 \neq 0$.

(a) $K_0 = 0$. Equation (47) is now an equation of the third order, and so by hypothesis all of its coefficients must vanish. Thus we have the eight integrability conditions,

(49) $K_i = 0, \qquad (i = 1, \cdots, 8).$

In this case it is impossible to express x_{uuuu} as a linear combination of derivatives of x of lower orders. But since all derivatives of x of the fifth order, and therefore all derivatives of higher orders, may be expressed as a linear combination of x_{uuuu}, x_{uuu}, x_{uvv}, x_{uu}, x_{uv}, x_{vv}, x_u, x_v, x, the integral surfaces of equations (41) will be immersed in a space S_8 determined by these nine independent points.

The relation,

$$K_0 = (A_0 + \gamma_2 + \ell)_v - (B_0 + \beta_1 + \delta_2 + a + b')_u = 0,$$

may be derived from

$$W = (x_{uuuu}, x_{uuu}, x_{uvv}, x_{uu}, x_{uv}, x_{vv}, x_u, x_v, x) \neq 0$$

in the same way that equation (37) was derived from the corresponding determinant for the fifth canonical form. Thus the function,

(50) $p = \log k W,$

has the property that

$$p_u = A_0 + \gamma_2 + \ell,$$

$$p_v = B_0 + \beta_1 + \delta_2 + a + b'.$$

(b) $K_0 \neq 0$. In this case equation (47) may be solved for x_{uuuu}, with the result

(51) $$x_{uuuu} = \alpha_1 x_{uuu} + \alpha_2 x_{uvv} + \alpha_3 x_{uu} + \alpha_4 x_{uv} + \alpha_5 x_{vv} + \alpha_6 x_u + \alpha_7 x_v + \alpha_8 x,$$

where

$$\alpha_i = -K_i / K_0, \qquad (i = 1, \cdots, 8).$$

Eight integrability conditions are found by differentiating

equation (51) with respect to v and subtracting the result from the second of equations (44) in which x_{uuuu} has been replaced by its value as given by equation (51). These conditions are

$$
\begin{aligned}
&\alpha_{1v} + \alpha_1\beta_1 + 4C^3\alpha_1\alpha_2 + \alpha_2\delta_1 + 4C^3\alpha_5 \\
&\qquad = \beta_1\alpha_1 + B_1, \\
&\alpha_{2v} + \alpha_1\beta_2 + 4C^3\alpha_2^2 + \alpha_2\delta_2 + \alpha_4 - 3C\alpha_5 \\
&\qquad = \beta_1\alpha_2 + B_2, \\
&\alpha_{3v} + \alpha_1\beta_3 + 4C^3\alpha_2\alpha_3 + \alpha_2\delta_3 + a\alpha_3 + a'\alpha_5 \\
&\qquad = \beta_1\alpha_3 + B_3, \\
&\alpha_{4v} + \alpha_1\beta_4 + 4C^3\alpha_2\alpha_4 + \alpha_2\delta_4 + k\alpha_3 + k'\alpha_5 + \alpha_6 \\
&\qquad = \beta_1\alpha_4 + B_4, \\
&\alpha_{5v} + \alpha_1\beta_5 + 4C^3\alpha_2\alpha_5 + \alpha_2\delta_5 + b\alpha_3 + b'\alpha_5 + \alpha_7 \\
&\qquad = \beta_1\alpha_5 + B_5, \\
&\alpha_{6v} + \alpha_1\beta_6 + 4C^3\alpha_2\alpha_6 + \alpha_2\delta_6 + l\alpha_3 + l'\alpha_5 \\
&\qquad = \beta_1\alpha_6 + B_6, \\
&\alpha_{7v} + \alpha_1\beta_7 + 4C^3\alpha_2\alpha_7 + \alpha_2\delta_7 + m\alpha_3 + m'\alpha_5 + \alpha_8 \\
&\qquad = \beta_1\alpha_7 + B_7, \\
&\alpha_{8v} + \alpha_1\beta_8 + 4C^3\alpha_2\alpha_8 + \alpha_2\delta_8 + d\alpha_3 + d'\alpha_5 \\
&\qquad = \beta_1\alpha_8 + B_8.
\end{aligned}
\tag{52}
$$

Eight further integrability conditions are found by differentiating equation (51) with respect to u and equating the value of x_{uuuuu} thus obtained to the value given by the first of equations (44) in which x_{uuuu} has been replaced by its value from equation (51). These conditions are

$$
\begin{aligned}
&\alpha_{1u} + \alpha_1^2 + \alpha_2 y_1 + \alpha_3 = A_0\alpha_1 + A_1,\\
&\alpha_{2u} + \alpha_1\alpha_2 + \alpha_2 y_2 + \alpha_5 = A_0\alpha_2 + A_2,\\
&\alpha_{3u} + \alpha_1\alpha_3 + \alpha_2 y_3 + a\alpha_4 + \alpha_6 = A_0\alpha_3 + A_3,\\
&\alpha_{4u} + \alpha_1\alpha_4 + \alpha_2 y_4 + \ell\alpha_4 + \alpha_7 = A_0\alpha_4 + A_4,\\
&\alpha_{5u} + \alpha_1\alpha_5 + \alpha_2 y_5 + b\alpha_4 = A_0\alpha_5 + A_5,\\
&\alpha_{6u} + \alpha_1\alpha_6 + \alpha_2 y_6 + \ell\alpha_4 + \alpha_8 = A_0\alpha_6 + A_6,\\
&\alpha_{7u} + \alpha_1\alpha_7 + \alpha_2 y_7 + m\alpha_4 = A_0\alpha_7 + A_7,\\
&\alpha_{8u} + \alpha_1\alpha_8 + \alpha_2 y_8 + d\alpha_4 = A_0\alpha_8 + A_8.
\end{aligned}
\tag{53}
$$

When these sixteen conditions are satisfied, all derivatives of x of the fourth order, and therefore all derivatives of higher orders, may be expressed as linear combinations of x_{uuu}, x_{uvv}, x_{uu}, x_{uv}, x_{vv}, x_u, x_v, x, and the integral surfaces of equations (41) will be immersed in a space S_7 determined by these eight independent points.

SECTION III

TRANSFORMATIONS OF THE EQUATIONS

In this section we shall consider those transformations of the pairs of equations in the fifth and sixth canonical forms, which do not change the form of these equations, or which yield equally simple canonical forms. The transformation of dependent variable,

$$x = \lambda(u,v)\,\bar{x}, \tag{54}$$

where λ is a scalar function not identically zero, represents merely a change in the proportionality factor of the homogeneous coordinates of the points of the integral surface S, and so leaves the surface invariant. We shall find that this transformation does not change the form of the equations in the fifth and sixth canonical forms, and that by suitably choosing the λ we may obtain reduced forms for these equations. The most general transformation of the independent variables u and v,

$$\bar{u} = U(u,v),\quad \bar{v} = V(u,v),\quad J = U_u V_v - U_v V_u \neq 0, \tag{55}$$

will in general change the form of the pair of differential equations by the introduction of all the derivatives of x of the third order. In the study of each of the canonical forms we shall determine the special types of this transformation which preserve the forms of the differential equations.

1. The Fifth Canonical Form. Applying transformation (54) to equations (26) of the fifth canonical form, we obtain a new system of equations of the same form in which the coefficients have the values:

$$
\begin{aligned}
&\bar{a} = a - \lambda_v/\lambda, \quad \bar{k} = k - 2\lambda_u/\lambda, \quad \bar{b} = b,\\
&\bar{l} = \tfrac{1}{\lambda}(l\lambda + 2a\lambda_u + k\lambda_v - 2\lambda_{uv}),\\
&\bar{m} = \tfrac{1}{\lambda}(m\lambda + k\lambda_u + 2b\lambda_v - \lambda_{uu}),\\
&\bar{d} = \tfrac{1}{\lambda}(d\lambda + a\lambda_{uu} + k\lambda_{uv} + b\lambda_{vv}\\
&\qquad + l\lambda_u + m\lambda_v - \lambda_{uuv}),
\end{aligned}
\tag{56}
$$

$$
\begin{aligned}
&\bar{D} = D, \quad \bar{a}' = a', \quad \bar{k}' = k' - 2\lambda_v/\lambda,\\
&\bar{b}' = \tfrac{1}{\lambda}(b'\lambda + 3D\lambda_v - \lambda_u),\\
&\bar{l}' = \tfrac{1}{\lambda}(l'\lambda + 2a'\lambda_u + k'\lambda_v - \lambda_{vv}),\\
&\bar{m}' = \tfrac{1}{\lambda}(m'\lambda + 3D\lambda_{vv} + k'\lambda_u + 2b'\lambda_v - 2\lambda_{uv}),\\
&\bar{d}' = \tfrac{1}{\lambda}(d'\lambda + D\lambda_{vvv} + a'\lambda_{uu} + k'\lambda_{uv} + b'\lambda_{vv}\\
&\qquad + l'\lambda_u + m'\lambda_v - \lambda_{uvv}).
\end{aligned}
$$

Applying transformation (55) to equations (26) we get a new system in which each equation involves all four of the derivatives of x of the third order. These may be solved for $x_{\bar{u}\bar{u}\bar{u}\bar{v}}$ and $x_{\bar{u}\bar{v}\bar{v}\bar{v}}$ in case the determinant Δ of the coefficients of these derivatives does not vanish, and in this section we shall consider only transformations for which this is true. Solving, we have $x_{\bar{u}\bar{u}\bar{u}\bar{v}}$ and $x_{\bar{u}\bar{v}\bar{v}\bar{v}}$ expressed as linear combinations

of $x_{\bar{u}\bar{u}\bar{u}}$, $x_{\bar{v}\bar{v}\bar{v}}$, and derivatives of x of lower orders. In order that these equations reduce to the form of equations (26) we must have the coefficients of $x_{\bar{u}\bar{u}\bar{u}}$ in both equations and the coefficient of $x_{\bar{v}\bar{v}\bar{v}}$ in the expression for $x_{\bar{u}\bar{u}\bar{v}}$ all equal to zero, while the coefficient of $x_{\bar{v}\bar{v}\bar{v}}$ in the expression for $x_{\bar{u}\bar{v}\bar{v}}$ must be different from zero. There are just two forms of the transformation (55) which satisfy these four conditions together with $\Delta \neq 0$. These are

(57) $\bar{u} = U(u), \qquad \bar{v} = V(v), \qquad J = U_u V_v \neq 0,$

and

(58) $\bar{u} = U(u), \qquad \bar{v} = V(u,v), \qquad J = U_u V_v \neq 0,$

where $V_u = 2D\, V_v.$

The effect of transformation (57) on equations (26) is to produce a new system of the same form in which the coefficients have the values:

$$\bar{a} = \frac{1}{V_v^3} a, \qquad \bar{k} = \frac{1}{U_u} k - \frac{U_{uu}}{U_u^2},$$

$$\bar{b} = \frac{V_v}{U_u^2} b, \qquad \bar{l} = \frac{1}{U_u V_v} l + \frac{U_{uu}}{U_u^2 V_v} a,$$

$$\bar{m} = \frac{1}{U_u^2} m + \frac{V_{vv}}{U_u^2 V_v} b, \qquad \bar{d} = \frac{1}{U_u^2 V_v} d,$$

(59) $$\bar{D} = \frac{V_v}{U_u} D, \qquad \bar{a}' = \frac{U_u}{V_v^2} a', \qquad \bar{k}' = \frac{1}{V_v} k' - \frac{V_{vv}}{V_v^2},$$

$$\bar{b}' = \frac{1}{U_u} b' + 3\frac{V_{vv}}{U_u V_v} D, \qquad \bar{l}' = \frac{1}{V_v^2} l' + \frac{U_{uu}}{U_u V_v^2} a',$$

$$\bar{m} = \frac{1}{U_u V_v} m' + \frac{V_{vvv}}{U_u V_v^2} D + \frac{V_{vv}}{U_u V_v^2} \ell',$$

$$\bar{d}' = \frac{1}{U_u V_v^2} d'.$$

Similarly on applying transformation (58) to equations (26) we get a new system of the same form in which the coefficients have the values:

$$\bar{a} = \frac{1}{V_v} (a - 4Da'),$$

$$\bar{k} = \frac{1}{U_u}\left[-4D_v - \frac{U_{uu}}{U_u} + 4D(a - 4Da') + (k - 4D\ell')\right],$$

$$\bar{\ell} = \frac{V_v}{U_u^2}\left[(4DD_v - 2D_u) + 4D^2(a - 4Da') + 2D(k - 4D\ell') + (\ell - 4D\ell')\right],$$

$$\bar{k} = \frac{1}{U_u V_v}(\ell - 4D\ell') + \frac{U_{uu}}{U_u^2 V_v}(a - 4Da'),$$

$$\bar{m} = \frac{1}{U_u^2}\left[-(4D_v^2 - 4DD_{vv} + 2D_{uv}) + (4DD_v + 2D_u)(a - 4Da') + 2D_v(k - 4D\ell') + 2D(\ell - 4D\ell') + (m - 4Dm')\right] + \frac{V_{vv}}{U_u^2 V_v}\left[(4DD_v - 2D_u) + 4D^2(a - 4Da') + 2D(k - 4D\ell') + (\ell - 4D\ell')\right],$$

$$\bar{d} = \frac{1}{U_u^2 V_v}(d - 4Dd'),$$

(60) $$\bar{D} = -\frac{V_v}{U_u} D, \qquad \bar{a}' = \frac{U_u}{V_v^2} a',$$

$$\bar{k}' = -\frac{V_{vv}}{V_v^2} + \frac{1}{V_v}(k' + 4Da'),$$

$$\bar{l}' = -3\frac{V_{vv}}{U_u V_v} D + \frac{1}{U_u}(-4D_v + 4a'D^2 + 2l'D + l'),$$

$$\bar{l}' = \frac{U_{uu}}{U_u V_v^2} a' + \frac{1}{V_v^2} l',$$

$$\bar{m}' = -\frac{V_{vvv}}{U_u V_v^2} D + \frac{V_{vv}}{U_u V_v^2}(-4D_v + 4a'D^2 + 2l'D + l')$$

$$+ \frac{1}{U_u V_v}[-2D_{vv} + a'(4DD_v + 2D_u) + 2l'D_v + 2l'D + m'],$$

$$\bar{d}' = \frac{1}{U_u V_v^2} d'.$$

The only coefficients of equations (26) which are invariants under transformations (54), (57), and (58) are D and a'. Geometrically $D = 0$ is the condition that equations (26) reduce to the fourth canonical form, and $a' = 0$ is the condition that distinguishes the two cases under the fifth canonical form. The coefficient b is also an invariant under transformations (54) and (57) which are the most general transformations preserving the parametric net on the integral surface S. Its vanishing therefore represents a geometric property of the parametric net; in fact, if $b = 0$ the surface S will admit a Laplace transformation of the form

$$X = x_v - a\,x.$$

We know that on an integral surface S of equations (26) there exists a single infinity of triples of one-parameter

families of curves, and that for just three of these triples not all of the families are distinct.[1] The differential equations of these singular triples are

$$du\, dv^2 = 0,$$

$$du^2\,(D\,du + dv) = 0,$$

$$du\,(2D\,du + dv)^2 = 0.$$

Since transformation (57) preserves the parametric net we may regard it as using the curves of the first singular triple as the parametric curves, while transformation (58) uses the curves of the third singular triple as parametric curves. There is also a transformation which will take the curves of the second singular triple for parametric curves. A simple form of this transformation is

$$(61)\qquad \bar{u} = U(u,v), \qquad \bar{v} = u, \qquad J = -U_v \neq 0,$$

where $U_u = D\,U_v$,

and a more general form in which $\bar{v}$ is taken as a general function of u may be obtained by taking the product of this transformation with transformation (57). The result of applying transformation (61) to equations (26) is

$$\begin{aligned} x_{\bar{u}\bar{u}\bar{v}} &= \bar{a}\,x_{\bar{u}\bar{u}} + \bar{k}\,x_{\bar{u}\bar{v}} + \bar{b}\,x_{\bar{v}\bar{v}} + \bar{l}\,x_{\bar{u}} \\ &\quad + \bar{m}\,x_{\bar{v}} + \bar{d}\,x, \\ (62)\qquad x_{\bar{u}\bar{v}\bar{v}} &= \bar{A}\,x_{\bar{u}\bar{u}\bar{u}} + \bar{a}'\,x_{\bar{u}\bar{u}} + \bar{k}'\,x_{\bar{u}\bar{v}} + \bar{b}'\,x_{\bar{v}\bar{v}} + \bar{l}'\,x_{\bar{u}} \end{aligned}$$

---

[1] E. P. Lane, loc. cit.

$$+ \bar{m}' x_v + \bar{d}' x ,$$

where the coefficients have the values:

$$\bar{a} = \frac{1}{U_v}\left(-2U_{uv} + 2D U_{vv} + a' D U_u + \ell' U_u + b' U_v\right),$$

$$\bar{\ell} = \frac{1}{U_v^2}\left(-U_{vv} + 2a' U_u + \ell' U_v\right), \qquad \bar{b} = \frac{1}{U_v^2} a',$$

$$\bar{l} = \frac{1}{U_v^2}\left(-U_{uvv} + D U_{vvv} + a' U_{uu} + \ell' U_{uv} + b' U_{vv} + \ell' U_u + m' U_v\right),$$

$$\bar{m} = \frac{1}{U_v^2} \ell', \qquad \bar{d} = \frac{1}{U_v^2} d',$$

$$\bar{A} = -D^2 U_v^2,$$

(63)
$$\bar{a}' = D U_{uv} - D_u U_v - 4 D^2 U_{vv} + D^2 U_v (a - 2D a') + U_u (\ell - 2D\ell') + U_v (b - 2D b'),$$

$$\bar{\ell}' = \frac{1}{U_v}\left[-2U_{uv} + 2D U_{vv} + 2U_u (a - 2D a') + U_v (\ell - 2D\ell')\right], \qquad \bar{b}' = \frac{1}{U_v}(a - 2D a'),$$

$$\bar{l}' = \frac{1}{U_v}\left[-U_{uuv} + 2D U_{uvv} - 2D^2 U_{vvv} + U_{uu}(a - 2D a') + U_{uv}(\ell - 2D\ell') + U_{vv}(b - 2D b') + U_u (\ell - 2D\ell') + U_v (m - 2D m')\right],$$

$$\bar{m}' = \frac{1}{U_v}(\ell - 2D\ell'), \qquad \bar{d}' = \frac{1}{U_v}(d - 2D d').$$

Either of the two systems (26) or (62) may be taken as the standard form for a pair of equations in the fifth canonical form.

From equations (56) we see that if λ be chosen as a solution of equations (26) we obtain a new system in which there are no terms involving x undifferentiated. Having so chosen λ we see from equations (59), (60), and (63) that none of the transformations of independent variables here considered will reintroduce such terms.

2. The Sixth Canonical Form. Applying transformation (54) to equations (41) of the sixth canonical form, we get a new system of the same form in which the coefficients have the values:

$$\bar{a} = a - \lambda_v/\lambda , \quad \bar{k} = k - 2\lambda_u/\lambda , \quad \bar{b} = b,$$

$$\bar{l} = \frac{1}{\lambda}\left(l\lambda - 2\lambda_{uv} + 2a\lambda_u + k\lambda_v \right),$$

$$\bar{m} = \frac{1}{\lambda}\left(m\lambda + k\lambda_u + 2b\lambda_v - \lambda_{uu} \right),$$

$$\bar{d} = \frac{1}{\lambda}\left(d\lambda + a\lambda_{uu} + k\lambda_{uv} + b\lambda_{vv} + l\lambda_u + m\lambda_v - \lambda_{uuv} \right),$$

(64)
$$\bar{C} = C , \quad \bar{a}' = a' + 12C^3\lambda_u/\lambda ,$$

$$\bar{k}' = k' - 6C\lambda_v/\lambda , \quad \bar{b}' = b' - 3C\lambda_u/\lambda - 3\lambda_v/\lambda,$$

$$\bar{l}' = \frac{1}{\lambda}\left(l'\lambda + 12C^3\lambda_{uu} - 3C\lambda_{vv} + 2a'\lambda_u + k'\lambda_v \right),$$

$$\bar{m}' = \frac{1}{\lambda}\left(m'\lambda - 6C\lambda_{uv} + k'\lambda_u + 2b'\lambda_v - 3\lambda_{vv} \right),$$

$$\bar{d}' = \frac{1}{\lambda}\left(d'\lambda + 4C^3\lambda_{uuu} - 3C\lambda_{uvv} + a'\lambda_{uu} + k'\lambda_{uv} + \ell'\lambda_{vv} + \ell'\lambda_u + m'\lambda_v - \lambda_{vvv}\right).$$

Applying transformation (55) to equations (41) and proceeding in a manner similar to that used in the preceding case, we find that there are four forms of transformation (55) which will yield new systems of the same form as equations (41). These are:

(65) $\bar{u} = U(u), \qquad \bar{v} = V(v), \qquad J = U_u V_v \neq 0,$

(66) $\bar{u} = U(u,v), \qquad \bar{v} = V(u,v),$

where $U_v = C\,U_u, \quad V_u = -(1/2C)\,V_v,$

$J = 3/2\;\; U_u V_v \neq 0,$

and

(67) $\bar{u} = U(u,v), \qquad \bar{v} = V(u,v$

where $U_v = -(2g + 3)\,C\,U_u, \quad V_u = (1/gC)\,V_v,$

$J = (-3g + 9/2)\;U_u V_v \neq 0;$

$g = \frac{1 \pm (-15)^{\frac{1}{2}}}{4}, \quad 2g^2 - g + 2 = 0.$

Applying transformation (65) to equations (41) we find that in the resulting system of equations the coefficients have the values:

$$\bar{a} = \frac{a}{V_v}, \quad \bar{k} = \frac{1}{U_u}k - \frac{U_{uu}}{U_u^2}, \quad \bar{\ell} = \frac{V_v}{U_u^2}\ell,$$

$$\bar{\ell} = \frac{1}{U_u V_v}\ell + \frac{U_{uu}}{U_u^2 V_v}a,$$

$$\bar{m} = \frac{1}{U_u^2}m + \frac{V_{vv}}{U_u^2 V_v}\ell, \quad \bar{d} = \frac{1}{U_u^2 V_v}d,$$

(68) $$\bar{C} = \frac{U_u}{V_v}C, \quad \bar{a}' = \frac{U_u^2}{V_v^3}a' + 12C^3\frac{U_u U_{uu}}{V_v^3},$$

$$\bar{k}' = \frac{U_u}{V_v^2} k' - 3C \frac{U_u V_{vv}}{V_v^3}, \quad \bar{b}' = \frac{1}{V_v} b' - 3 \frac{V_{vv}}{V_v^2},$$

$$\bar{l}' = \frac{U_u}{V_v^3} l' + 4C^3 \frac{U_{uuu}}{V_v^3} + \frac{U_{uu}}{V_v^3} a',$$

$$\bar{m}' = \frac{1}{V_v^2} m' - \frac{V_{vvv}}{V_v^3} + \frac{V_{vv}}{V_v^3} b', \quad \bar{d}' = \frac{1}{V_v^3} d'.$$

Similarly the effect of transformation (66) on equations (41) is to produce a system of the same form in which the coefficients have the values:

$$\bar{a} = \frac{2}{27 C^2 V_v} (-12 C^2 C_u - 6 C C_v + a' + C k' + C^2 b'),$$

$$\bar{k} = \frac{-1}{27 C^3 U_u} (27 C^3 \frac{U_{uu}}{U_u} + 3 C^2 C_u + 15 C C_v + 2 a' - C k' - 4 C^2 b'),$$

$$\bar{b} = \frac{V_v}{54 C^4 U_u^2} (-12 C^2 C_u - 6 C C_v + a' - 2 C k' + 4 C^2 b'),$$

$$\bar{l} = \frac{2}{27 C^2 U_u V_v} \left[(-12 C^2 C_u - 6 C C_v + a' + C k' + C^2 b') \frac{U_{uu}}{U_u} + (-4 C^2 C_{uu} - 4 C C_u^2 - 2 C_u C_v - 4 C C_{uv} - C_{vv} + C_u k' + C C_u b' + C_v b' + l' + C m') \right],$$

$$\bar{m} = \frac{2}{27 C^2 U_u^2} \left[(-\frac{3 C_v}{2C} - 3 C_u + \frac{a'}{4C^2} - \frac{k'}{2C} + b') \frac{V_{vv}}{V_v} + (-\frac{C_{vv}}{C} + \frac{3 C_v^2}{2 C^2} + \frac{4 C_u C_v}{C} - C_{uv} + 2 C C_{uu} - 4 C_u^2 - \frac{a' C_v}{4 C^3} + \frac{a' C_u}{2 C^2} + \frac{k' C_v}{2 C^2} - \frac{l'}{2C} + m') \right],$$

$$\bar{d} = \frac{2}{27 C^2 U_u^2 V_v} d',$$

(69)
$$\bar{C} = -\frac{U_u}{V_v} C,$$

$$\bar{a}' = 4C^2 \frac{U_u^2}{V_v^3}\left(-3C\frac{U_{uu}}{U_u} - 2C_u + a + kC + bC^2\right),$$

$$\bar{k}' = 4C^2 \frac{U_u}{V_v^2}\left(\frac{3}{4C}\frac{V_{vv}}{V_v} - \frac{3C_v}{4C^2} + \frac{C_u}{2C} - \frac{a}{C} + \frac{k}{2} + 2Cb\right),$$

$$\bar{b}' = -2C\frac{1}{V_v}\left(\frac{3}{2C}\frac{V_{vv}}{V_v} - \frac{3C_v}{2C^2} + \frac{C_u}{C} - \frac{a}{2C} + k - 2Cb\right),$$

$$\bar{l}' = 4C^2 \frac{U_u}{V_v^3}\left[-C\frac{U_{uuu}}{U_u} + (-2C_u + a + Ck + C^2 b)\frac{U_{uu}}{U_u} - C_{uu} + kC_u + b(CC_u + C_v) + l + C_m\right],$$

$$\bar{m}' = 4C^2 \frac{1}{V_v^2}\left[-\frac{1}{4C^2}\frac{V_{vvv}}{V_v} + \left(\frac{3C_v}{4C^3} - \frac{C_u}{2C^2} + \frac{a}{4C^2} - \frac{k}{2C} + b\right)\frac{V_{vv}}{V_v} + \frac{C_{vv}}{4C^3} - \frac{3C_v^2}{4C^4} - \frac{C_{uv}}{2C^2} + \frac{C_u C_v}{C^3} - \frac{aC_v}{4C^3} + \frac{aC_u}{2C^2} + \frac{kC_v}{2C^2} - \frac{l}{2C} + m\right],$$

$$\bar{d}' = 4C^2 \frac{1}{V_v^3} d.$$

Applying the transformations (67) to equations (41) we get a new system of the form

(70)
$$\begin{aligned} x_{\bar{u}\bar{u}\bar{v}} &= \bar{T}, \\ x_{\bar{v}\bar{v}\bar{v}} &= -4C^3 \frac{U_u^3}{V_v^3} x_{\bar{u}\bar{u}\bar{u}} + 3C\frac{U_u}{V_v} x_{\bar{u}\bar{v}\bar{v}} + \bar{T}', \end{aligned}$$

where $\bar{T}$ and $\bar{T}'$ denote linear combinations of the derivatives of

x with respect to $\bar{u}$ and $\bar{v}$ of the second and lower orders. If needed the values of the coefficients in $\overline{T}$ and $\overline{T}'$ may be calculated as in the preceding cases. Using for $\overline{C}$ the value

$$\overline{C} = - C \frac{U_u}{V_v},$$

we note that equations (70) are of precisely the same form as equations (41).

On an integral surface S of equations (41) of the sixth canonical form there is an infinity of triples of one-parameter families of curves such that in just four of the triples not all of the families are distinct.[2] The differential equations of these four singular triples are

$$du\ dv^2 = 0,$$

$$(du + C\ dv)\ (du - 2C\ dv)^2 = 0,$$

$$[du + gC\ dv]^2\ [du - (2g + 3)\ C\ dv] = 0.$$

Hence the four forms of transformation (55) which preserve the form of equations (41) are those transformations which take as parametric curves the families of these four singular triples.

The only coefficient of equations (41) which is an invariant under all the transformations considered is C. When it is zero equations (41) reduce to the second canonical form. The coefficient b is an invariant under transformations (54) and (65) which preserve the parametric net on S. Its vanishing is the condition that the surface S shall admit a Laplace transformation of the form

$$X = x_v - a\ x.$$

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[2] Ibid.

From equations (64) we see that if λ be chosen as a solution of equations (41) the transformed system will involve no terms in x undifferentiated. Having made this choice of λ no transformation of the independent variables will reintroduce such terms.

SECTION IV

THE LOCAL COORDINATES

The study of the geometry of a surface in the neighborhood of a point is simplified by the introduction of a system of local coordinates. In this section we shall define such systems of coordinates for each of the three types of integral surfaces we are considering, which lie in spaces of a definite number of dimensions; that is, for integral surfaces of the fifth canonical form with $a' \neq 0$, and for the integral surfaces of the two possible cases under the sixth canonical form. In each case we shall use these coordinates to study the properties of the system of hyperplanes that have contact of various orders with the integral surfaces.

1. The Fifth Canonical Form with $a' \neq 0$. In the space S_7, which is the ambient space of the integral surfaces of equations (26) of the fifth canonical form with $a' \neq 0$ and subject to the integrability conditions (36), the eight points x, x_u, x_v, x_{uu}, x_{uv}, x_{vv}, x_{uuu}, x_{vvv} are independent and are the vertices of a non-degenerate pyramid, which we shall use as the local pyramid of reference. Since x_{uuv}, x_{uvv}, and all derivatives of x of higher orders may be expressed uniquely as linear combinations of the eight derivatives of lower orders including x, every solution X of equations (26) may be expressed in the form

$$X = x_1 x + x_2 x_u + x_3 x_v + x_4 x_{uu} + x_5 x_{uv} + x_6 x_{vv}$$

$$+ x_7 x_{uuu} + x_8 x_{vvv}.$$

By suitably choosing the unit point we may use

$$(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8)$$

as the local coordinates of the point P_X.

Consider now an integral surface S of equations (26), a point P_x on S with parameter values (u,v), and let P_X be a point on S near P_x with parameter values $(u + \Delta u, v + \Delta v)$. Then by the use of Taylor's development we may express the local coordinates of P_X as power series expansions in Δu and Δv. These expansions through fourth degree terms in Δu and Δv are:

$$\begin{aligned}
x_1 &= 1 + \tfrac{1}{2}(a\,\Delta u^2 \Delta v + a'\Delta u\,\Delta v^2) + \tfrac{1}{24}(\alpha_1 \Delta u^4 \\
&\quad + 4\beta_1 \Delta u^3 \Delta v + 6\gamma_1 \Delta u^2 \Delta v^2 + 4\delta_1 \Delta u \Delta v^3 + \epsilon_1 \Delta v^4) + \cdots, \\
x_2 &= \Delta u + \tfrac{1}{2}(b\,\Delta u^2 \Delta v + b'\Delta u\,\Delta v^2) + \tfrac{1}{24}(\alpha_2 \Delta u^4 \\
&\quad + 4\beta_2 \Delta u^3 \Delta v + 6\gamma_2 \Delta u^2 \Delta v^2 + 4\delta_2 \Delta u \Delta v^3 + \epsilon_2 \Delta v^4) + \cdots, \\
x_3 &= \Delta v + \tfrac{1}{2}(c\,\Delta u^2 \Delta v + c'\Delta u\,\Delta v^2) + \tfrac{1}{24}(\alpha_3 \Delta u^4 \\
&\quad + 4\beta_3 \Delta u^3 \Delta v + 6\gamma_3 \Delta u^2 \Delta v^2 + 4\delta_3 \Delta u \Delta v^3 + \epsilon_3 \Delta v^4) + \cdots, \\
x_4 &= \tfrac{1}{2}\Delta u^2 + \tfrac{1}{2}(d\,\Delta u^2 \Delta v + d'\Delta u\,\Delta v^2) + \tfrac{1}{24}(\alpha_4 \Delta u^4 \\
&\quad + 4\beta_4 \Delta u^3 \Delta v + 6\gamma_4 \Delta u^2 \Delta v^2 + 4\delta_4 \Delta u \Delta v^3 + \epsilon_4 \Delta v^4) + \cdots, \\
x_5 &= \Delta u \Delta v + \tfrac{1}{2}(e\,\Delta u^2 \Delta v + e'\Delta u\,\Delta v^2) + \tfrac{1}{24}(\alpha_5 \Delta u^4 \\
&\quad + 4\beta_5 \Delta u^3 \Delta v + 6\gamma_5 \Delta u^2 \Delta v^2 + 4\delta_5 \Delta u \Delta v^3 + \epsilon_5 \Delta v^4) + \cdots, \\
x_6 &= \tfrac{1}{2}\Delta v^2 + \tfrac{1}{2}(f\,\Delta u^2 \Delta v + f'\Delta u\,\Delta v^2) + \tfrac{1}{24}(\alpha_6 \Delta u^4
\end{aligned} \tag{71}$$

$$+ 4\beta_5 \Delta u^3 \Delta v + 6\gamma_5 \Delta u^2 \Delta v^2 + 4\delta_5 \Delta u \Delta v^3 + \epsilon_5 \Delta v^4) + \cdots,$$

$$x_7 = \frac{1}{6} \Delta u^3 + \frac{1}{24} (\alpha_1 \Delta u^4 + 4\beta_1 \Delta u^3 \Delta v + 4\delta_1 \Delta u \Delta v^3 + \epsilon_1 \Delta v^4) + \cdots,$$

$$x_8 = \frac{1}{2} D \Delta u \Delta v^2 + \frac{1}{6} \Delta v^3 + \frac{1}{24} (\alpha_2 \Delta u^4 + 4\beta_2 \Delta u^3 \Delta v + 6\gamma_2 \Delta u^2 \Delta v^2 + 4\delta_2 \Delta u \Delta v^3 + \epsilon_2 \Delta v^4) + \cdots.$$

Of the ∞^7 hyperplanes in the space S_7 there are ∞^6 passing through P_x represented by the equation

$$\sum_{i=2}^{8} \xi_i x_i = 0$$

Those that have first order contact with S at P_x are found by requiring that this equation be satisfied identically through first degree terms in Δu and Δv by the expansions (71). This imposes the conditions,

$$\xi_2 = \xi_3 = 0.$$

Thus there are ∞^4 hyperplanes through P_x which have first order contact with S at P_x. They all contain the tangent plane to S at P_x, and are represented by the equation

$$\sum_{i=4}^{8} \xi_i x_i = 0.$$

The hyperplanes with second order contact with S at P_x are found by requiring that this equation be satisfied identically by the expansions (71) through second degree terms in Δu and Δv. This imposes the additional conditions,

$$\Sigma_4 = \Sigma_5 = \Sigma_6 = 0.$$

Hence there is a pencil of hyperplanes,

$$\Sigma_7 x_7 + \Sigma_8 x_8 = 0,$$

each of which has second order contact with S at P_x and so contains the osculating space S(2,0) to S at P_x. Each of these hyperplanes cuts S in a curve with a triple point at P_x, the directions of the principal tangents at the triple point being given by

$$\Sigma_7 du^3 + 3\Sigma_8 D\,du\,dv^2 + \Sigma_8 dv^3 = 0. \tag{72}$$

As in the general case, we note here that each triple of directions defined by this equation is apolar to every triple of the system

$$\lambda D\,du^3 + \lambda\,du^2 dv - \mu\,du\,dv^2 = 0 \tag{73}$$

where λ and μ are parameters. This latter system is the special form, for the equations of the fifth canonical form, of the general system of covariant triples of directions used in classifying the pairs of differential equations of the third order. The fact that it has only three singular triples each of the first kind is characteristic of the equations of the fifth canonical form.[1] By equating the discriminant of the system (72) to zero we find that there are just three hyper-

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[1] E. P. Lane, loc. cit.

planes for which the three directions defined by equation (72) are not all distinct. These are $x_7 = 0$, for which the three directions coincide in the direction of the v curves, i.e. $du^3 = 0$; $x_8 = 0$, for which the three directions are

$$dv^2 (3 D du + dv) = 0;$$

and $4 D^3 x_7 - x_8 = 0$, for which the three directions are

$$(2 D du + dv)^2 (D du - dv) = 0.$$

The three directions, $du = 0$, $dv = 0$, $2 D du + dv = 0$, which are the only ones in which two directions of the same triple of equation (72) can coincide are also the directions of the double families of the system (73).[2]

There is no hyperplane that meets S at P_x with contact of the third order.

2. The Sixth Canonical Form with $K_o \neq 0$. The integral surfaces of equations (41) of the sixth canonical form, with $K_o \neq 0$ and subject to the integrability conditions (52) and (53), lie in a space S_7 which is determined by the eight independent points x, x_u, x_v, x_{uu}, x_{uv}, x_{vv}, x_{uuu}, x_{uvv}. We shall take these points as the vertices of the pyramid of reference and define the local coordinates as in the preceding case. If P_X, with parameter values $(u + \Delta u, v + \Delta v)$, is a point on an integral surface S of equations (41) near P_x, with parameter values (u,v), then the power series expansions of the local coordinates of P_X are found to be:

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[2] Ibid.

$$\chi_1 = 1 + \tfrac{1}{6}(3\alpha\,\Delta u^2\Delta v + \alpha'\Delta v^3) + \tfrac{1}{24}\left[\alpha_1\,\Delta u^4 + 4\beta_1\,\Delta u^3\Delta v + 6\gamma_1\,\Delta u^2\Delta v^2 + 4(4C^3\alpha_1 + \delta_1)\,\Delta u\,\Delta v^3 + (-12C^4\alpha_1 + \epsilon_1)\,\Delta v^4\right] + \cdots,$$

$$\chi_2 = \Delta u + \tfrac{1}{6}(3\ell\,\Delta u^2\Delta v + \ell'\Delta v^3) + \tfrac{1}{24}\left[\alpha_2\,\Delta u^4 + 4\beta_2\,\Delta u^3\Delta v + 6\gamma_2\,\Delta u^2\Delta v^2 + 4(4C^3\alpha_2 + \delta_2)\,\Delta u\,\Delta v^3 + (-12C^4\alpha_2 + \epsilon_2)\,\Delta v^4\right] + \cdots,$$

$$\chi_3 = \Delta v + \tfrac{1}{6}(3m\,\Delta u^2\Delta v + m'\Delta v^3) + \tfrac{1}{24}\left[\alpha_3\,\Delta u^4 + 4\beta_3\,\Delta u^3\Delta v + 6\gamma_3\,\Delta u^2\Delta v^2 + 4(4C^3\alpha_3 + \delta_3)\,\Delta u\,\Delta v^3 + (-12C^4\alpha_3 + \epsilon_3)\,\Delta v^4\right] + \cdots,$$

$$\chi_4 = \tfrac{1}{2}\Delta u^2 + \tfrac{1}{6}(3a\,\Delta u^2\Delta v + a'\Delta v^3) + \tfrac{1}{24}\left[\alpha_4\,\Delta u^4 + 4\beta_4\,\Delta u^3\Delta v + 6\gamma_4\,\Delta u^2\Delta v^2 + 4(4C^3\alpha_4 + \delta_4)\,\Delta u\,\Delta v^3 + (-12C^4\alpha_4 + \epsilon_4)\,\Delta v^4\right] + \cdots,$$

(74)

$$\chi_5 = \Delta u\,\Delta v + \tfrac{1}{6}(3k\,\Delta u^2\Delta v + k'\Delta v^3) + \tfrac{1}{24}\left[\alpha_5\,\Delta u^4 + 4\beta_5\,\Delta u^3\Delta v + 6\gamma_5\,\Delta u^2\Delta v^2 + 4(4C^3\alpha_5 + \delta_5)\,\Delta u\,\Delta v^3 + (-12C^4\alpha_5 + \epsilon_5)\,\Delta v^4\right] + \cdots,$$

$$\chi_6 = \tfrac{1}{2}\Delta v^2 + \tfrac{1}{6}(3b\,\Delta u^2\Delta v + b'\Delta v^3) + \tfrac{1}{24}\left[\alpha_6\,\Delta u^4 + 4\beta_6\,\Delta u^3\Delta v + 6\gamma_6\,\Delta u^2\Delta v^2 + 4(4C^3\alpha_6 + \delta_6)\,\Delta u\,\Delta v^3 + (-12C^4\alpha_6 + \epsilon_6)\,\Delta v^4\right] + \cdots,$$

$$\chi_7 = \tfrac{1}{6}\Delta u^3 + \tfrac{2}{3}C^3\Delta v^3 + \tfrac{1}{24}\left[\alpha_7\,\Delta u^4 + 4\beta_7\,\Delta u^3\Delta v + 6\gamma_7\,\Delta u^2\Delta v^2 + 4(4C^3\alpha_7 + \delta_7)\,\Delta u\,\Delta v^3 + (-12C^4\alpha_7 + \epsilon_7)\,\Delta v^4\right] + \cdots,$$

$$\chi_8 = \tfrac{1}{2}\Delta u\,\Delta v^2 - \tfrac{1}{2}C\,\Delta v^3 + \tfrac{1}{24}\left[\alpha_8\,\Delta u^4 + 4\beta_8\,\Delta u^3\Delta v + 6\gamma_8\,\Delta u^2\Delta v^2 + 4(4C^3\alpha_8 + \delta_8)\,\Delta u\,\Delta v^3 + (-12C^4\alpha_8 + \epsilon_8)\,\Delta v^4\right] + \cdots$$

As in the preceding case there are ∞^6 hyperplanes in the space S_7 through P_x, ∞^4 of these contain the tangent plane to S and have first order contact with S at P_x, and a pencil of these hyperplanes contain the osculating space S(2,0) to S at P_x and have second order contact with S there. The equation of this pencil is

$$\xi_7 x_7 + \xi_8 x_8 = 0.$$

Each of these hyperplanes cuts S in a curve with a triple point at P_x, the directions of the principal tangents at the triple point being given by

$$\xi_7 du^3 + 3\xi_8 du\, dv^2 + (4C^3\xi_7 - 3C\xi_8)\, dv^3 = 0. \tag{75}$$

As before each triple of directions of this system is apolar to every triple of the system

$$\lambda du^3 - 3\lambda C du^2 dv + 3\mu du dv^2 + 4\lambda C^3 dv^3 = 0, \tag{76}$$

where λ and μ are parameters, which characterizes the integral surfaces of a pair of differential equations of the sixth canonical form.[3] There are four hyperplanes for which the triples of directions defined by equation (75) are not all distinct. These are $x_8 = 0$, for which the directions are

$$dv^2 (du - C\, dv) = 0;$$

$x_7 - 4C^3 x_8 = 0$, for which the directions are

$$(du + 4C\, dv)(du - 2C\, dv)^2 = 0;$$

and $\epsilon x_7 + (2 - g) C^2 x_8 = 0$, for which the directions are

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[3] Ibid.

$$(du + g\,C\,dv)^2\ (du - 2\,g\,C\,dv) = 0,$$

where g is defined in connection with equations (67). The four directions, $dv = 0$, $du - 2\,C\,dv = 0$, $du + g\,C\,dv = 0$, which are the only ones in which two directions of a single triple of system (75) may coincide are also the directions of the double families of the system (76).[4]

There is no hyperplane that has third order contact with S at P_x.

3. The Sixth Canonical Form with $K_o = 0$. The integral surfaces of equations (41) of the sixth canonical form, with $K_o = 0$ and subject to the integrability conditions (49), lie in a space S_8 determined by the nine independent points x, x_u, x_v, x_{uu}, x_{uv}, x_{vv}, x_{uuu}, x_{uvv}, x_{uuuu}. These points are the vertices of a non-degenerate pyramid which we shall choose as the local pyramid of reference. The coordinates of any point, referred to the original system of coordinates, may be written

$$x_1\,x + x_2\,x_u + x_3\,x_v + x_4\,x_{uu} + x_5\,x_{uv} + x_6\,x_{vv} + x_7\,x_{uuu} + x_8\,x_{uvv} + x_9\,x_{uuuu},$$

and by suitably choosing the unit point,

$$(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9)$$

become the local coordinates of the point. If P_x, with parameter values (u,v), is a point on S, and P_X is a neighboring point on S, with parameter values $(u + \Delta u, v + \Delta v)$, then the local coordinates of P_X expressed as power series expansions in Δu and Δv are:

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[4] Ibid.

$$X_1 = 1 + \tfrac{1}{6}(3d\,\Delta u^2\Delta v + d'\,\Delta v^3) + \tfrac{1}{24}(4\beta_8\,\Delta u^3\Delta v + 6\gamma_8\,\Delta u^2\Delta v^2 + 4\delta_8\,\Delta u\,\Delta v^3 + \epsilon_8\,\Delta v^4) + \cdots,$$

$$X_2 = \Delta u + \tfrac{1}{6}(3\ell\,\Delta u^2\Delta v + \ell'\Delta v^3) + \tfrac{1}{24}(4\beta_6\,\Delta u^3\Delta v + 6\gamma_6\,\Delta u^2\Delta v^2 + 4\delta_6\,\Delta u\,\Delta v^3 + \epsilon_6\,\Delta v^4) + \cdots,$$

$$X_3 = \Delta v + \tfrac{1}{6}(3m\,\Delta u^2\Delta v + m'\Delta v^3) + \tfrac{1}{24}(4\beta_7\,\Delta u^3\Delta v + 6\gamma_7\,\Delta u^2\Delta v^2 + 4\delta_7\,\Delta u\,\Delta v^3 + \epsilon_7\,\Delta v^4) + \cdots,$$

$$X_4 = \tfrac{1}{2}\Delta u^2 + \tfrac{1}{6}(3a\,\Delta u^2\Delta v + a'\Delta v^3) + \tfrac{1}{24}(4\beta_3\,\Delta u^3\Delta v + 6\gamma_3\,\Delta u^2\Delta v^2 + 4\delta_3\,\Delta u\,\Delta v^3 + \epsilon_3\,\Delta v^4) + \cdots, \tag{77}$$

$$X_5 = \Delta u\,\Delta v + \tfrac{1}{6}(3\ell\,\Delta u^2\Delta v + \ell'\Delta v^3) + \tfrac{1}{24}(4\beta_4\,\Delta u^3\Delta v + 6\gamma_4\,\Delta u^2\Delta v^2 + 4\delta_4\,\Delta u\,\Delta v^3 + \epsilon_4\,\Delta v^4) + \cdots,$$

$$X_6 = \tfrac{1}{2}\Delta v^2 + \tfrac{1}{6}(3b\,\Delta u^2\Delta v + b'\Delta v^3) + \tfrac{1}{24}(4\beta_5\,\Delta u^3\Delta v + 6\gamma_5\,\Delta u^2\Delta v^2 + 4\delta_5\,\Delta u\,\Delta v^3 + \epsilon_5\,\Delta v^4) + \cdots,$$

$$X_7 = \tfrac{1}{6}(\Delta u^3 + 4C^3\Delta v^3) + \tfrac{1}{24}(4\beta_1\,\Delta u^3\Delta v + 6\gamma_1\,\Delta u^2\Delta v^2 + 4\delta_1\,\Delta u\,\Delta v^3 + \epsilon_1\,\Delta v^4) + \cdots,$$

$$X_8 = \tfrac{1}{6}(3\Delta u\,\Delta v^2 - 3C\,\Delta v^3) + \tfrac{1}{24}(4\beta_2\,\Delta u^3\Delta v + 6\gamma_2\,\Delta u^2\Delta v^2 + 4\delta_2\,\Delta u\,\Delta v^3 + \epsilon_2\,\Delta v^4) + \cdots,$$

$$X_9 = \tfrac{1}{24}(\Delta u^4 + 16\,C^3\,\Delta u\,\Delta v^3 - 12C^4\,\Delta v^4) + \cdots.$$

Following the same method as in the two preceding cases we find that of the ∞^7 hyperplanes in the S_8 and through P_x ∞^5 contain the tangent plane to S at P_x and have first order contact with S there, and ∞^2 contain the osculat-space S(2,0) to S at P_x, and have with S at P_x second order contact. The equation of this double infinity of hyperplanes is

$$\xi_7 x_7 + \xi_8 x_8 + \xi_9 x_9 = 0.$$

Each of these hyperplanes cuts S in a curve with a triple point at P_x, the directions of the principal tangents at the triple point being given by equation (75). In finding those triples for which not all of the directions are distinct the algebraic computation is identical with that for the case $K_o \neq 0$, but since the ambient space here is of higher dimensionality the geometric interpretation is different. There are four spaces S_6 with equations: $x_8 = 0$, $x_9 = 0$; $x_7 - 4\,C^3 x_8 = 0$, $x_9 = 0$; $2\,x_7 + (2 - g)\,C^2 x_8 = 0$, $x_9 = 0$, such that any hyperplane containing one of these spaces will cut S in a curve whose triple point tangents at P_x are not all distinct. All the hyperplanes on one of these spaces S_6 will cut S in curves having the same triple of directions for its principal tangents, the triple being the same as for the corresponding hyperplane in the preceding case where $K_o \neq 0$.

There is just one hyperplane, $x_9 = 0$, that has third order contact with S at P_x. This hyperplane cuts S in a curve with a quadruple point at P_x, the directions of the principal

tangents being

$$du^4 + 16\,C^3\,du\,dv^3 - 12\,C^4\,dv^4 = 0,$$

or

$$\{du - [-1 \pm 3^{\frac{1}{2}}]\,C\,dv\}\,\{du - [1 \pm (-5)^{\frac{1}{2}}]\,C\,dv\} = 0.$$

SECTION V

OSCULATING SPACES TO CURVES ON THE INTEGRAL SURFACES

Segre,[1] by the use of hyperplane coordinates, has considered the osculating spaces of one, two, and three dimensions to curves lying on a surface in a general space S_n. In this section we shall consider these spaces for curves on those integral surfaces which we are studying, which lie in spaces of a definite number of dimensions, that is, for the integral surfaces of equations in the fifth canonical form with $a' \neq 0$, and for the integral surfaces under the two cases of the sixth canonical form, and shall determine the equations of these spaces in terms of point coordinates. In each case we shall extend the theory to include the osculating spaces of four dimensions. In the two cases where the integral surfaces lie in spaces of seven dimensions it becomes impossible because of the limited dimensionality of the ambient space to define certain figures in the usual way as loci of osculating spaces, and in their place we shall consider certain quadric hypercones defined as the envelopes of linear osculants to curves C as these curves vary in a prescribed manner.

1. The Fifth Canonical Form with $a' \neq 0$. Consider an integral surface S of equations (26) of the fifth canonical

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[1] C. Segre, Su una classe superficie degl' iperspazi, legata colle equazione lineari alle derivate parziali di 2° ordine. Atti della R. Acad. delle Scienze di Torino, vol. 42 (1906-1907), p. 1047 ff.

form with $a' \neq 0$, and a point P_x on S. Consider also a curve C on S through P_x which belongs to the family $dv - \lambda\, du = 0$. Regard u as the parameter along C, and denote by accents total derivatives with respect to u. Let P_X be a point near P_x on C with parameter value $u + \Delta u$. Then the local coordinates of P_X expressed as power series in Δu are:

$$x_1 = 1 + \tfrac{1}{2}(a\lambda + a'\lambda^2)\Delta u^3 + \tfrac{1}{24}(\alpha_1 + 4\beta_1\lambda + 6\gamma_1\lambda^2 + 4\delta_1\lambda^3 + \epsilon_1\lambda^4 + 6a\lambda' + 12a'\lambda\lambda')\Delta u^4 + \cdots,$$

$$x_2 = \Delta u + \tfrac{1}{2}(b\lambda + b'\lambda^2)\Delta u^3 + \tfrac{1}{24}(\alpha_2 + 4\beta_2\lambda + 6\gamma_2\lambda^2 + 4\delta_2\lambda^3 + \epsilon_2\lambda^4 + 6b\lambda' + 12b'\lambda\lambda')\Delta u^4 + \cdots,$$

$$x_3 = \lambda\Delta u + \tfrac{1}{2}\lambda'\Delta u^2 + \tfrac{1}{6}(3m\lambda + 3m'\lambda^2 + \lambda'')\Delta u^3$$

(78)
$$+ \tfrac{1}{24}(\alpha_3 + 4\beta_3\lambda + 6\gamma_3\lambda^2 + 4\delta_3\lambda^3 + \epsilon_3\lambda^4 + 6m\lambda' + 12m'\lambda\lambda' + \lambda''')\Delta u^4 + \cdots,$$

$$x_4 = \tfrac{1}{2}\Delta u^2 + \tfrac{1}{2}(a\lambda + a'\lambda^2)\Delta u^3 + \tfrac{1}{24}(\alpha_4 + 4\beta_4\lambda + 6\gamma_4\lambda^2 + 4\delta_4\lambda^3 + \epsilon_4\lambda^4 + 6a\lambda' + 12a'\lambda\lambda')\Delta u^4 + \cdots,$$

$$x_5 = \lambda\Delta u^2 + \tfrac{1}{2}(b\lambda + b'\lambda^2 + \lambda')\Delta u^3 + \tfrac{1}{24}(\alpha_5 + 4\beta_5\lambda + 6\gamma_5\lambda^2 + 4\delta_5\lambda^3 + \epsilon_5\lambda^4 + 6b\lambda' + 12b'\lambda\lambda' + 4\lambda'')\Delta u^4 + \cdots,$$

$$x_6 = \tfrac{1}{2}\lambda^2\Delta u^2 + \tfrac{1}{2}(b\lambda + b'\lambda^2 + \lambda\lambda')\Delta u^3 + \tfrac{1}{24}(\alpha_6 + 4\beta_6\lambda + 6\gamma_6\lambda^2 + 4\delta_6\lambda^3 + \epsilon_6\lambda^4 + 6b\lambda' + 12b'\lambda\lambda' + 3\lambda^2 + 4\lambda\lambda'')\Delta u^4 + \cdots,$$

$$x_7 = \frac{1}{6}\Delta u^3 + \frac{1}{24}(\alpha_1 + 4\beta_1\lambda + 4\delta_1\lambda^3 + \epsilon_1\lambda^4)\Delta u^4 + \cdots,$$

$$x_8 = \frac{1}{6}(3D\lambda^2 + \lambda^3)\Delta u^3 + \frac{1}{24}(\alpha_2 + 4\beta_2\lambda + 6\gamma_2\lambda^2 + 4\delta_2\lambda^3 + \epsilon_2\lambda^4 + 12D\lambda\lambda' + 6\lambda^2\lambda')\Delta u^4 + \cdots.$$

The tangent line to C at P_x is found by considering the ∞^6 hyperplanes of the space S_7 through P_x,

$$\sum_{i=2}^{8} \xi_i x_i = 0, \tag{79}$$

and requiring that this equation be satisfied identically through terms in Δu by the expansions (78). This imposes the condition,

$$\xi_2 + \lambda \xi_3 = 0,$$

and shows that there are ∞^5 hyperplanes having first order contact with C at P_x and therefore passing through the tangent line. The equations of six independent ones are

$$x_3 - \lambda x_2 = x_4 = x_5 = x_6 = x_7 = x_8 = 0,$$

which may be taken as the equations of the tangent line. The locus of these tangent lines as λ varies is the tangent plane to S at P_x with equations

$$x_4 = x_5 = x_6 = x_7 = x_8 = 0.$$

The osculating plane to C at P_x is found by requiring that equation (79) be satisfied identically through terms in Δu^2 by the expansions (78). This imposes the additional condition,

$$\xi_4 + \lambda' \xi_3 + 2\lambda \xi_5 + \lambda^2 \xi_6 = 0.$$

Thus there are ∞^4 hyperplanes having second order contact with C at P_x and therefore containing the osculating plane. The equations of five independent ones are:

$$\begin{aligned} x_3 - \lambda x_2 - \lambda' x_4 &= 0, \\ x_5 - 2\lambda x_4 &= 0, \\ x_6 - \lambda^2 x_4 &= 0, \\ x_7 &= 0, \\ x_8 &= 0, \end{aligned}$$

which may be taken as the equations of the osculating plane to C at P_x. The locus of the osculating planes to all curves on S through P_x with the same tangent line is a space S_3 with equations,

$$x_5 - 2\lambda x_4 = x_6 - \lambda^2 x_4 = x_7 = x_8 = 0.$$

The locus of the osculating planes to all curves on S through P_x is found by eliminating λ from these equations. The result is

$$4 x_4 x_6 - x_5^2 = x_7 = x_8 = 0.$$

This is the equation of a quadric hypercone V_4^2, lying in the osculating space S(2,0) to S at P_x. The geometric properties of this cone have been discussed by Segre.[2] It is called the cone of Del Pezzo.

The osculating space S_3 to C at P_x is found by requiring that equation (79) be satisfied identically by the expansions (78) through terms in Δu^3. This imposes a third condi-

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[2] C. Segre, loc. cit., p. 1051.

tion upon the coefficients of equation (79),

$$E\,\mathfrak{f}_3 + F\,\mathfrak{f}_5 + G\,\mathfrak{f}_6 + \mathfrak{f}_7 + H\,\mathfrak{f}_8 = 0,$$

where

$$E = 3m\lambda + 3m'\lambda^2 + \lambda'' - 3\ell\lambda^2 - 3\ell'\lambda^3 - 3a\lambda\lambda' - 3a'\lambda^2\lambda',$$

$$F = 3k\lambda + 3k'\lambda^2 + 3\lambda' - 6a\lambda^2 - 6a'\lambda^3,$$

$$G = 3\ell\lambda + 3\ell'\lambda^2 + 3\lambda\lambda' - 3a\lambda^3 - 3a'\lambda^4,$$

$$H = 3D\lambda^2 + \lambda^3.$$

Thus there are ∞^3 hyperplanes that have third order contact with C at P_x and so pass through the osculating space S_3 to C. Four independent ones have the equations:

$$x_3 - \lambda x_2 - \lambda' x_4 - E\,x_7 = 0,$$

$$x_5 - 2\lambda x_4 - F\,x_7 = 0,$$

$$x_6 - \lambda^2 x_4 - G\,x_7 = 0,$$

$$x_8 - H\,x_7 = 0,$$

which may be taken as the equations of the osculating space S_3. The locus of the osculating spaces S_3 to all curves C that have at P_x the same element E_2, that is the same tangent line and osculating plane, is a space S_4 whose three equations are found by eliminating λ'' from the equations of the osculating space S_3. Since λ'' appears only in E we merely omit the first equation and use the three others. The locus of the osculating spaces S_3 to all curves C that have at P_x the same element E_1 is found by eliminating λ' from these three equa-

tions. The result is

$$\lambda x_5 - x_6 - \lambda^2 x_4 + (G - \lambda F)\, x_7 = 0,$$

$$x_8 - H\, x_7 = 0.$$

The locus is thus seen to be a space S_5. Eliminating λ from these two equations we get the equation of the locus of the osculating spaces S_3 to all curves C through P_x. This equation is of fifth degree and represents a quintic hypercone V_6^5 in the space S_7 which is the ambient space of S. The equation may be written in determinant form

$$\begin{vmatrix} q & r & s & 0 & 0 \\ 0 & q & r & s & 0 \\ 0 & 0 & q & r & s \\ x_7 & 3Dx_7 & 0 & -x_8 & 0 \\ 0 & x_7 & 3Dx_7 & 0 & -x_8 \end{vmatrix} = 0,$$

where

$$q = x_4 + 3(h - b' - D\,K)x_7\,,$$

$$r = -x_5 - 3\,b\,x_7 - 3\,a'\,x_8\,,$$

$$s = x_6 + K\,x_8\,,$$

$$K = 3\,h' - 3\,a + 9\,D\,a'.$$

Segre has discussed some of the properties of this locus,[3] and has shown geometrically that it must be a quintic cone V_6^5 in the osculating space S(3,0) which in our case becomes the ambient space S_7 of S.

The osculating space S_4 to C at P_x is found by requir-

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[3] C. Segre, loc. cit., pp. 1053-4.

ing that equation (79) be satisfied identically by the expansions (78) through terms in Δu^4. This imposes a further condition on the coefficients of equation (79),

$$M\xi_3 + N\xi_5 + Q\xi_6 + R\xi_8 = 0,$$

where

$$\begin{aligned}
M = {} & \alpha_7 + 4\beta_7\lambda + 6\gamma_7\lambda^2 + 4\delta_7\lambda^3 + \epsilon_7\lambda^4 + 6m\lambda' + 12m'\lambda\lambda' \\
& + \lambda''' - \lambda(\alpha_6 + 4\beta_6\lambda + 6\gamma_6\lambda^2 + 4\delta_6\lambda^3 + \epsilon_6\lambda^4 + 6\ell\lambda' \\
& + 12\ell'\lambda\lambda') - \lambda'(\alpha_3 + 4\beta_3\lambda + 6\gamma_3\lambda^2 + 4\delta_3\lambda^3 \\
& + \epsilon_3\lambda^4 + 6a\lambda' + 12a'\lambda\lambda') - E(\alpha_1 + 4\beta_1\lambda \\
& + 4\delta_1\lambda^3 + \epsilon_1\lambda^4), \\
N = {} & \alpha_4 + 4\beta_4\lambda + 6\gamma_4\lambda^2 + 4\delta_4\lambda^3 + \epsilon_4\lambda^4 + 6k\lambda' \\
& + 12k'\lambda\lambda' + 4\lambda'' - 2\lambda(\alpha_3 + 4\beta_3\lambda + 6\gamma_3\lambda^2 + 4\delta_3\lambda^3 \\
& + \epsilon_3\lambda^4 + 6a\lambda' + 12a'\lambda\lambda') - F(\alpha_1 + 4\beta_1\lambda + 4\delta_1\lambda^3 + \epsilon_1\lambda^4), \\
Q = {} & \alpha_5 + 4\beta_5\lambda + 6\gamma_5\lambda^2 + 4\delta_5\lambda^3 + \epsilon_5\lambda^4 + 6\ell\lambda' \\
& + 12\ell'\lambda\lambda' + 3\lambda'^2 + 4\lambda\lambda'' - \lambda^2(\alpha_3 + 4\beta_3\lambda + 6\gamma_3\lambda^2 \\
& + 4\delta_3\lambda^3 + \epsilon_3\lambda^4 + 6a\lambda' + 12a'\lambda\lambda') - G(\alpha_1 + 4\beta_1\lambda \\
& + 4\delta_1\lambda^3 + \epsilon_1\lambda^4), \\
R = {} & \alpha_2 + 4\beta_2\lambda + 6\gamma_2\lambda^2 + 4\delta_2\lambda^3 + \epsilon_2\lambda^4 + 12D\lambda\lambda' \\
& + 6\lambda^2\lambda' - H(\alpha_1 + 4\beta_1\lambda + 4\delta_1\lambda^3 + \epsilon_1\lambda^4).
\end{aligned}$$

If for any set of values of u, v, and λ it happens that M, N, Q, and R all vanish, the osculating space S_4 to C at that point is indeterminate, and the point will be a singular point on C. We shall not consider such points. Hence at P not all of M, N, Q, and R vanish and we shall assume that $R \neq 0$. Thus we have a double infinity of hyperplanes with contact of the

fourth order with C at P_x and so passing through the osculating space S_4 to C at that point. Three independent ones have the equations

$$R\,x_3 - R\,\lambda\,x_2 - R\,\lambda'\,x_4 - ER\,x_7 - M(x_8 - H\,x_7) = 0,$$
$$R\,x_5 - 2\,R\,\lambda\,x_4 - F\,R\,x_7 - N(x_8 - H\,x_7) = 0,$$
$$R\,x_6 - R\,\lambda^2\,x_4 - G\,R\,x_7 - Q(x_8 - H\,x_7) = 0,$$

which may be taken as the equations of the osculating space S_4 to C at P_x. The equations of the locus of the osculating spaces S_4 to all curves C that have at P_x the same element E_3 is found by eliminating λ''' from these equations. Since λ''' enters only in M we merely omit the first of these equations and use the two others. Hence the locus is a space S_5. The equation of the locus of the osculating spaces S_4 to all curves C that have at P_x the same element E_2 is found by eliminating λ'' from these equations. Since λ'' enters only in N and Q, and since $Q - \lambda\,N$ is free from λ'', we have on eliminating

$$(80)\qquad R\,\lambda\,x_5 - R\,x_6 - R\,\lambda^2\,x_4 + R\,(G - \lambda\,F)\,x_7 - (Q - \lambda\,N)\,(H\,x_7 - x_8) = 0.$$

The locus is thus seen to be a hyperplane.

The locus of the osculating spaces S_4 to all curves C through P_x that have the same element E_1, or to all the curves C through P_x, will be an infinity of hyperplanes which will fill the ambient space S_7 and will therefore have no particular interest. But for each direction λ at P_x there will be, corresponding to all possible values of λ', an infinity of the hyperplanes just considered which are the loci of the oscula-

ting spaces S_4 corresponding to fixed values of λ and λ'. Let us consider the hypersurface enveloped by these hyperplanes as λ' varies. To find its equation we must differentiate equation (80) with respect to λ' and eliminate λ' between equation (80) and the equation obtained by differentiation. For this purpose write

$$\mathcal{R} = \overline{\mathcal{R}} + (12D\lambda + 6\lambda^2)\lambda',$$

$$Q - \lambda N = T + 6U\lambda' + 3\lambda'^2,$$

where

$$\overline{\mathcal{R}} = \alpha_2 + 4\beta_2\lambda + (6\gamma_2 - 3D\alpha_1)\lambda^2 + (4\delta_2 - \alpha_1 - 12D\beta_1)\lambda^3$$
$$+ (\epsilon_2 - 4\beta_1)\lambda^4 - 12D\delta_1\lambda^5 - (3D\epsilon_1 + 4\delta_1)\lambda^6 - \epsilon_1\lambda^7,$$

$$T = \alpha_5 + (4\beta_5 - \alpha_4)\lambda + (6\gamma_5 - 4\beta_4)\lambda^2 + (4\delta_5 - 6\gamma_4)\lambda^3$$
$$+ (\epsilon_5 - 4\delta_4)\lambda^4 - \epsilon_4\lambda^5 + \lambda^2(\alpha_3 + 4\beta_3\lambda + 6\gamma_3\lambda^2 + 4\delta_3\lambda^3$$
$$+ \epsilon_3\lambda^4) - (G - \lambda F)(\alpha_1 + 4\beta_1\lambda + 4\delta_1\lambda^3 + \epsilon_1\lambda^4),$$

$$U = b + (2b' - b)\lambda - (2b' - a)\lambda^2 + 2a'\lambda^3.$$

G - λ F and H do not involve λ'. Differentiating and eliminating λ' we have

(81) $$(Hx_7 - x_8)\{\overline{\mathcal{R}}[\lambda x_5 - x_6 - \lambda^2 x_4 + (G - \lambda F)x_7] - T(Hx_7 - x_8)\}$$
$$+ 3\{(2D\lambda + \lambda^2)[\lambda x_5 - x_6 - \lambda^2 x_4 + (G - \lambda F)x_7] - U(Hx_7 - x_8)\}^2 = 0.$$

Hence for each direction λ at P_x there is a quadric hypercone, a V_6^2 with equation (81), enveloped by the infinity of hyperplanes, corresponding to the infinity of values of λ', which are the loci of the osculating spaces S_4 to those curves C

through P_x that have at P_x the same element E_2. Equation (81) may also be written in the form

$$3(2D\lambda+\lambda^2)^2\left[\lambda x_5 - x_6 - \lambda^2 x_4 + (G-\lambda F)x_7\right]^2$$
$$+\left[\overline{R} - 6U(2D\lambda+\lambda^2)\right](Hx_7 - x_8)\left[\lambda x_5 - x_6 - \lambda^2 x_4 + (G-\lambda F)x_7\right]$$
$$+ (3U^2 - T)(Hx_7 - x_8)^2 = 0.$$

The left member of this equation is a quadratic form in the two linear expressions $[\lambda x_5 - x_6 - \lambda^2 x_4 + (G - \lambda F) x_7]$ and $(H x_7 - x_8)$. If its discriminant is zero, that is if

$$\overline{R}^2 + 12(T - \overline{R}U)(2D\lambda + \lambda^2) = 0,$$

the left member of the equation becomes the square of a linear expression and the cone degenerates into a single hyperplane counted twice.

On substituting into equation (81) the expansions (71) we find that the equation is satisfied identically through terms of the third degree in Δu and Δv. The cone therefore has contact with S of at least the third order. The fourth degree terms, on replacing Δu and Δv by du and dv, are

$$(2D\lambda + \lambda^2)(dv - \lambda\, du)^4.$$

Hence if $\lambda(2D + \lambda) \neq 0$, the cone will have contact with S of exactly the third order, and will meet S in a curve with a quadruple point at P_x such that all four of the principal tangents coincide in the direction $dv - \lambda\, du = 0$. But if $\lambda(2D + \lambda) = 0$, the cone will have contact of at least the fourth order. We examine in detail the two possible cases.

(a) $\lambda = 0$. Here $dv = 0$ and we have the cone corre-

sponding to the direction of the u-curve. The equation of the cone becomes

$$\alpha_2 x_6 x_8 + (3b^2 - \alpha_5) x_8^2 = 0,$$

which shows that the cone breaks up into planes. On substituting into this equation the expansions (71) we have for the fifth degree terms

$$\alpha_2 \, dv^4 \, (3 D \, du + dv).$$

We shall see below (Section VI) that $\alpha_2 = 0$ is the condition that the u-curves on S be quasi-asymptotics of the type $\gamma_{2,4}$. Hence if the u-curves on S are not quasi-asymptotics of the type $\gamma_{2,4}$, the cone corresponding to the direction of the u-curve will have contact with S of exactly the fourth order. It will cut S in a curve with a quintuple point at P_x, four of the principal tangents coinciding with the tangent to the u-curve and the fifth lying in the direction $dv + 3 D \, du = 0$. If the u-curves on S are quasi-asymptotics of the type $\gamma_{2,4}$, the cone corresponding to the direction of the u-curve will have contact of at least the fifth order with S at P_x.

(b) $\lambda = -2D$. In this case the equation of the cone is

$$\bar{K} \left[-2D x_5 - x_6 - 4D^2 x_4 + (G + 2DF) x_7 \right] \left[4D^3 x_7 - x_8 \right]$$

$$+ (3U^2 - T)(4D^3 x_7 - x_8)^2 = 0,$$

where in the values for $\bar{K}$, G, F, U, T, we have replaced λ by $-2D$. On substituting the expansions (71) into this equation we have for the terms of fifth degree

$$\bar{R}\,(dv + 2\,D\,du)^4\,(dv - D\,du).$$

Hence if the value of $\bar{R}$ is different from zero the cone corresponding to the direction $dv + 2\,D\,du = 0$ will have contact of exactly the fourth order with S. It will cut S in a curve with a quintuple point at P_x with four of the principal tangents coinciding in the direction $dv + 2\,D\,du = 0$ and the fifth lying in the direction $dv - D\,du = 0$. If $\bar{R}$ is zero the cone will have contact of at least the fifth order with S at P_x.

By taking the differential equation of C in the form $dv - \lambda\,du = 0$, we have excluded the case in which C passes through P_x in the direction of the v-curve. To consider these curves let C have the equation

$$u = u(v), \qquad u'\big/_{P_x} = 0,$$

where accents denote total derivatives with respect to v. Then v becomes the parameter on C, and u'', u''', u^{iv} are the parameters which determine respectively the osculating spaces S_2, S_3, S_4 to C at P_x. The local coordinates of a point P_X, with parameter value $v + \Delta v$, on C near P_x are:

$$\begin{aligned}
x_1 &= 1 + \tfrac{1}{24}\,(\epsilon_8 + 6\,a'u'')\,\Delta v^4 + \cdots,\\
x_2 &= \tfrac{1}{2}\,u''\Delta v^2 + \tfrac{1}{6}\,u'''\Delta v^3 + \tfrac{1}{24}\,(\epsilon_6 + 6\,\ell'u'' + u^{iv})\,\Delta v^4 + \cdots,\\
x_3 &= \Delta v + \tfrac{1}{24}\,(\epsilon_7 + 6\,m'u'')\,\Delta v^4 + \cdots,\\
x_4 &= \tfrac{1}{24}\,(\epsilon_3 + 6\,a'u'' + 3u''^2)\,\Delta v^4 + \cdots,\\
x_5 &= \tfrac{1}{2}\,u''\,\Delta v^3 + \tfrac{1}{24}\,(\epsilon_4 + 6\,\ell'u'' + 4u''')\,\Delta v^4 + \cdots,\\
x_6 &= \tfrac{1}{2}\,\Delta v^2 + \tfrac{1}{24}\,(\epsilon_5 + 6\,\ell'u'')\,\Delta v^4 + \cdots,
\end{aligned} \tag{82}$$

$$x_7 = \frac{1}{24}\epsilon_1 \Delta v^4 + \cdots,$$

$$x_8 = \frac{1}{6}\Delta v^3 + \frac{1}{24}(\epsilon_2 + 6D u'')\Delta v^4 + \cdots.$$

The theory concerning the osculating spaces S_1, S_2, S_3 is similar to that for the general case and needs no further comment. Three independent hyperplanes through the osculating space S_4 are

$$x_2 - u'' x_6 - u''' x_8 + \frac{D^2}{a'}(\epsilon_6 + 6\ell' u'' + u^{IV} - \epsilon_5 u'' - 6\ell' u''^2 - \epsilon_2 u''' - 6D u'' u''')x_7 = 0,$$

$$x_4 + \frac{D^2}{a'}(\epsilon_3 + 6a' u'' + 3u''^2)x_7 = 0,$$

$$x_5 - 3u'' x_8 + \frac{D^2}{a'}(\epsilon_4 + 6\ell' u'' + 4u''' - 3\epsilon_2 u'' - 18D u''^2)x_7 = 0.$$

The locus of the osculating spaces S_4 to all curves C that have at P_x the same element E_3 is a space S_5 whose equations are the last two of these three equations. The locus of the osculating spaces S_4 to all curves C that have at P_x the same element E_2 is the hyperplane with equation

$$x_4 + \frac{D^2}{a'}(\epsilon_3 + 6a' u'' + 3u''^2)x_7 = 0.$$

The cone V_6^2, corresponding to that given by equation (81), which is the envelope of these hyperplanes as u'' varies, has the equation

$$\epsilon_1 x_4 x_7 + (3a'^2 - \epsilon_3)x_7^2 = 0.$$

The cone is thus seen to degenerate into two planes. Substi-

tuting the expansions (71) into this equation we find that the cone has contact of at least the fourth order with S. The fifth degree terms in du and dv are

$$\epsilon_1 \, du^5.$$

Since in this case where $a' \neq 0$, we have $\epsilon_1 \neq 0$, the cone has contact with S of exactly the fourth order, and meets S in a curve with a quintuple point at P_x such that all five of the principal tangents coincide with the tangent to the v-curve.

Thus we conclude that the cones given by equation (81) for all the directions about P_x will in general have contact with S of the third order, and only in case the cone corresponds to one of the three directions $dv = 0$, $du = 0$, $dv + 2D\,du = 0$ will the contact be of fourth or higher order. We note that these are the directions of the double families in the three singular triples of one-parameter families of curves on S.[4]

2. The Sixth Canonical Form with $K_0 \neq 0$. Consider an integral surface S of equations (41) of the sixth canonical form with $K_0 \neq 0$, and a point P_x on S. Let C be the curve of the family $dv - \lambda\,du = 0$ on S through P_x and let P_X be a point on C near P_x with parameter value $u + \Delta u$. Then the local coordinates of P_X are:

$$x_1 = 1 + \tfrac{1}{6}(3a\lambda + a'\lambda^3)\Delta u^3 + \tfrac{1}{24}[\alpha_8 + 4\beta_8\lambda + 6\gamma_8\lambda^2 + 4(\delta_8 + 4c^3\alpha_8)\lambda^3 + (\epsilon_8 - 12c^4\alpha_8)\lambda^4 + 6a\lambda' + 6a'\lambda^2\lambda']\Delta u^4 + \cdots,$$

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[4] E. P. Lane, loc. cit.

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\begin{aligned}
x_2 &= \Delta u + \tfrac{1}{6}(3\ell\lambda + \ell'\lambda^3)\Delta u^3 + \tfrac{1}{24}\big[\alpha_6 + 4\beta_6\lambda \\
&\quad + 6\gamma_6\lambda^2 + 4(\delta_6 + 4C^3\alpha_6)\lambda^3 + (\epsilon_6 - 12C^4\alpha_6)\lambda^4 \\
&\quad + 6\ell\lambda' + 6\ell'\lambda^2\lambda'\big]\Delta u^4 + \cdots, \\
x_3 &= \lambda\Delta u + \tfrac{1}{2}\lambda'\Delta u^2 + \tfrac{1}{6}(3m\lambda + m'\lambda^3 + \lambda'')\Delta u^3 \\
&\quad + \tfrac{1}{24}\big[\alpha_7 + 4\beta_7\lambda + 6\gamma_7\lambda^2 + 4(\delta_7 + 4C^3\alpha_7)\lambda^3 \\
&\quad + (\epsilon_7 - 12C^4\alpha_7)\lambda^4 + 6m\lambda' + 6m'\lambda^2\lambda' \\
&\quad + \lambda'''\big]\Delta u^4 + \cdots, \\
x_4 &= \tfrac{1}{2}\Delta u^2 + \tfrac{1}{6}(3a\lambda + a'\lambda^3)\Delta u^3 + \tfrac{1}{24}\big[\alpha_3 + 4\beta_3\lambda \\
&\quad + 6\gamma_3\lambda^2 + 4(\delta_3 + 4C^3\alpha_3)\lambda^3 + (\epsilon_3 - 12C^4\alpha_3)\lambda^4 \\
&\quad + 6a\lambda' + 6a'\lambda^2\lambda'\big]\Delta u^4 + \cdots, \qquad (83) \\
x_5 &= \lambda\Delta u^2 + \tfrac{1}{6}(3k\lambda + k'\lambda^3 + 3\lambda')\Delta u^3 + \tfrac{1}{24}\big[\alpha_4 + 4\beta_4\lambda \\
&\quad + 6\gamma_4\lambda^2 + 4(\delta_4 + 4C^3\alpha_4)\lambda^3 + (\epsilon_4 - 12C^4\alpha_4)\lambda^4 \\
&\quad + 6k\lambda' + 6k'\lambda^2\lambda' + 4\lambda''\big]\Delta u^4 + \cdots, \\
x_6 &= \tfrac{1}{2}\lambda^2\Delta u^2 + \tfrac{1}{6}(3k\lambda + k'\lambda^3 + 3\lambda\lambda')\Delta u^3 + \tfrac{1}{24}\big[\alpha_5 \\
&\quad + 4\beta_5\lambda + 6\gamma_5\lambda^2 + 4(\delta_5 + 4C^3\alpha_5)\lambda^3 \\
&\quad + (\epsilon_5 - 12C^4\alpha_5)\lambda^4 + 6k\lambda' + 6k'\lambda^2\lambda' \\
&\quad + 3\lambda'^2 + 4\lambda\lambda''\big]\Delta u^4 + \cdots, \\
x_7 &= \tfrac{1}{6}(1 + 4C^3\lambda^3)\Delta u^3 + \tfrac{1}{24}\big[\alpha_1 + 4\beta_1\lambda \\
&\quad + 6\gamma_1\lambda^2 + 4(\delta_1 + 4C^3\alpha_1)\lambda^3 + (\epsilon_1 - 12C^4\alpha_1)\lambda^4 \\
&\quad + 24C^3\lambda^2\lambda'\big]\Delta u^4 + \cdots, \\
x_8 &= \tfrac{1}{6}(3\lambda^2 - 3C\lambda^3)\Delta u^3 + \tfrac{1}{24}\big[\alpha_2 + 4\beta_2\lambda \\
&\quad + 6\gamma_2\lambda^2 + 4(\delta_2 + 4C^3\alpha_2)\lambda^3 + (\epsilon_2 - 12C^4\alpha_2)\lambda^4 \\
&\quad + 12\lambda\lambda' - 18C\lambda^2\lambda'\big]\Delta u^4 + \cdots.
\end{aligned}
$$

The equations of the osculating spaces S_1 and S_2 to C at P_x and of the spaces derived from them are identical with

the corresponding equations in the preceding case. As in that case, the equation of the ∞^4 hyperplanes through the osculating space S_2 to C is

$$(84) \qquad \begin{aligned} &\xi_3 (x_3 - \lambda x_2 - \lambda' x_4) + \xi_5 (x_5 - 2\lambda x_4) \\ &\xi_6 (x_6 - \lambda^2 x_4) + \xi_7 x_7 + \xi_8 x_8 = 0. \end{aligned}$$

To find the hyperplanes through the osculating space S_3 to C we must require that equation (84) be satisfied identically by the expansions (83) through terms in Δu^3. This imposes the condition,

$$E\xi_3 + F\xi_5 + G\xi_6 + H\xi_7 + J\xi_8 = 0,$$

where

$$\begin{aligned} E &= 3m\lambda + m'\lambda^3 + \lambda'' - 3\ell\lambda^2 - \ell'\lambda^4 - 3a\lambda\lambda' - a'\lambda^3\lambda', \\ F &= 3k\lambda + k'\lambda^3 + 3\lambda' - 6a\lambda^2 - 2a'\lambda^4, \\ G &= 3b\lambda + b'\lambda^3 + 3\lambda\lambda' - 3a\lambda^3 - a'\lambda^5, \\ H &= 1 + 4c'\lambda^3, \\ J &= 3\lambda^2 - 3c\lambda^3. \end{aligned}$$

Thus there are ∞^3 hyperplanes that have contact of the third order with C and so pass through the osculating space S_3 to C at P_x. There is evidently no value for λ that will make H and J vanish simultaneously. If $H \neq 0$, the equations of four independent hyperplanes through the osculating space S_3 to C are

$$(85) \qquad \begin{aligned} H(x_3 - \lambda x_2 - \lambda' x_4) - E x_7 &= 0, \\ H(x_5 - 2\lambda x_4) - F x_7 &= 0, \\ H(x_6 - \lambda^2 x_4) - G x_7 &= 0, \end{aligned}$$

$$H x_8 - J x_7 = 0.$$

If $J \neq 0$ the equations of four such hyperplanes are

$$\begin{aligned} J(x_3 - \lambda x_2 - \lambda' x_4) - E x_8 &= 0, \\ J(x_5 - 2\lambda x_4) - F x_8 &= 0, \\ J(x_6 - \lambda^2 x_4) - G x_8 &= 0, \\ J x_7 - H x_8 &= 0. \end{aligned} \tag{86}$$

The locus of the osculating spaces S_3 to all curves C through P_x with the same element E_2 is a space S_4 whose equations are found by omitting the first equation in either of the two sets of equations (85) and (86). The equations of the locus of the osculating spaces S_3 to all curves C that have at P_x the same element E_1 is found by eliminating λ' from these equations. Since λ' appears only in F and G, and since $G - \lambda F$ is free from λ' we have for the desired equations, using equations (85),

$$\begin{aligned} H(\lambda x_5 - x_6 - \lambda^2 x_4) + (G - \lambda F) x_7 &= 0, \\ H x_8 - J x_7 &= 0, \end{aligned}$$

and using equations (86),

$$\begin{aligned} J(\lambda x_5 - x_6 - \lambda^2 x_4) + (G - \lambda F) x_8 &= 0, \\ J x_7 - H x_8 &= 0. \end{aligned}$$

The locus is seen to be a space S_5. The equation of the locus of the osculating spaces S_3 to all curves C through P_x is found by eliminating λ from either of these pairs of equations. Writing the second pair as polynomials in λ it is possible to combine in such a way that we may divide by λ^2,

since $J \neq 0$, $\lambda \neq 0$, and obtain two equations of third degree in λ. Applying Sylvester's Dialytic Method of Elimination we get a determinant of the sixth order which may be reduced to the form:

$$\begin{vmatrix} \mathcal{A}+C\mathcal{B}+C^2\mathcal{b}-3C^3x_4 & \mathcal{B} & \mathcal{b} & -3x_4 & 0 & 0 \\ C(\mathcal{A}+C\mathcal{B}+C^2\mathcal{b}-3C^3x_4) & \mathcal{A} & \mathcal{B} & \mathcal{b} & -3x_4 & 0 \\ C^2(\mathcal{A}+C\mathcal{B}+C^2\mathcal{b}-3C^3x_4) & 0 & \mathcal{A} & \mathcal{B} & \mathcal{b} & -3x_4 \\ 5C^3x_8 & -3x_7 & 0 & x_8 & 0 & 0 \\ 5C^4x_8 & (4C^3x_8+3Cx_7) & -3x_7 & 0 & x_8 & 0 \\ 5C^5x_8 & 0 & (4C^3x_8+3Cx_7) & -3x_7 & 0 & x_8 \end{vmatrix} = 0,$$

where

$$\mathcal{A} = 3Cx_4 + 9C\mathcal{l}x_7 + (12C^3\mathcal{l}+a')x_8,$$

$$\mathcal{B} = -3x_4 - 3Cx_5 - 9(C\mathcal{b}+\mathcal{l})x_7 - (12C^3\mathcal{b}+\mathcal{l}')x_8,$$

$$\mathcal{b} = 3x_5 + 3Cx_6 + 9\mathcal{b}x_7 + (3a+\mathcal{b}')x_8,$$

$$\mathcal{A}+C\mathcal{B}+C^2\mathcal{b}-3C^3x_6 = (12C^3\mathcal{l}+a'-12C^4\mathcal{b} - C\mathcal{l}'+3C^2a+C^2\mathcal{b}')x_8.$$

Resolving the factor x_8 from the first column of the determinant, we have left an equation of fifth degree which is the equation of the desired locus. Using the first pair of equations we write them as polynomials in λ of the fifth and third degrees. Applying Sylvester's Dialytic Method of Elimi-

nation we get a determinant of the eighth order which may be reduced so that x_7 will be a factor of each of the first three columns. Removing these we have the desired equation of the fifth degree for the locus.

In finding the hyperplanes through the osculating space S_4 to C we consider first the case in which $H \neq 0$. We must require that equation (84) be satisfied identically through terms in Δu^4 by the expansions (83). This imposes a further condition on the ξ's,

$$M\xi_3 + N\xi_5 + Q\xi_6 + R\xi_7 = 0,$$

where

$$\begin{aligned}
M = {} & H[\alpha_7 + 4\beta_7\lambda + 6\gamma_7\lambda^2 + 4(\delta_7 + 4C^3\alpha_7)\lambda^3 \\
& + (\epsilon_7 - 12C^4\alpha_7)\lambda^4 + 6m\lambda' + 6m'\lambda^2\lambda' + \lambda'''] \\
& - H\lambda[\alpha_6 + 4\beta_6\lambda + 6\gamma_6\lambda^2 + 4(\delta_6 + 4C^3\alpha_6)\lambda^3 \\
& + (\epsilon_6 - 12C^4\alpha_6)\lambda^4 + 6l\lambda' + 6l'\lambda^2\lambda'] \\
& - H\lambda'[\alpha_3 + 4\beta_3\lambda + 6\gamma_3\lambda^2 + 4(\delta_3 + 4C^3\alpha_3)\lambda^3 \\
& + (\epsilon_3 - 12C^4\alpha_3)\lambda^4 + 6a\lambda' + 6a'\lambda^2\lambda'] \\
& - E[\alpha_1 + 4\beta_1\lambda + 6\gamma_1\lambda^2 + 4(\delta_1 + 4C^3\alpha_1)\lambda^3 \\
& + (\epsilon_1 - 12C^4\alpha_1)\lambda^4 + 24C^3\lambda^2\lambda'],
\end{aligned}$$

$$\begin{aligned}
N = {} & H[\alpha_4 + 4\beta_4\lambda + 6\gamma_4\lambda^2 + 4(\delta_4 + 4C^3\alpha_4)\lambda^3 \\
& + (\epsilon_4 - 12C^4\alpha_4)\lambda^4 + 6k\lambda' + 6k'\lambda^2\lambda' + 4\lambda''] \\
& - 2H\lambda[\alpha_3 + 4\beta_3\lambda + 6\gamma_3\lambda^2 + 4(\delta_3 + 4C^3\alpha_3)\lambda^3 \\
& + (\epsilon_3 - 12C^4\alpha_3)\lambda^4 + 6a\lambda' + 6a'\lambda^2\lambda']
\end{aligned}$$

$$- F[\alpha_1 + 4\beta_1\lambda + 6\gamma_1\lambda^2 + 4(\delta_1 + 4C^3\alpha_1)\lambda^3 + (\epsilon_1 - 12C^4\alpha_1)\lambda^4 + 24C^3\lambda^2\lambda'],$$

$$Q = H[\alpha_5 + 4\beta_5\lambda + 6\gamma_5\lambda^2 + 4(\delta_5 + 4C^3\alpha_5)\lambda^3 + (\epsilon_5 - 12C^4\alpha_5)\lambda^4 + 6b\lambda' + 6b'\lambda^2\lambda' + 3\lambda'^2 + 4\lambda\lambda'']$$

$$- H\lambda^2[\alpha_3 + 4\beta_3\lambda + 6\gamma_3\lambda^2 + 4(\delta_3 + 4C^3\alpha_3)\lambda^3 + (\epsilon_3 - 12C^4\alpha_3)\lambda^4 + 6a\lambda' + 6a'\lambda^2\lambda']$$

$$- G[\alpha_1 + 4\beta_1\lambda + 6\gamma_1\lambda^2 + 4(\delta_1 + 4C^3\alpha_1)\lambda^3 + (\epsilon_1 - 12C^4\alpha_1)\lambda^4 + 24C^3\lambda^2\lambda'],$$

$$R = H[\alpha_2 + 4\beta_2\lambda + 6\gamma_2\lambda^2 + 4(\delta_2 + 4C^3\alpha_2)\lambda^3 + (\epsilon_2 - 12C^4\alpha_2)\lambda^4 + 12\lambda\lambda' - 18C\lambda^2\lambda']$$

$$- J[\alpha_1 + 4\beta_1\lambda + 6\gamma_1\lambda^2 + 4(\delta_1 + 4C^3\alpha_1)\lambda^3 + (\epsilon_1 - 12C^4\alpha_1)\lambda^4 + 24C^3\lambda^2\lambda'].$$

As in the preceding case we suppose that P_x is an ordinary point on C so that not all M, N, Q, R vanish, and in particular we suppose $R \neq 0$. Thus there are ∞^2 hyperplanes with contact of the fourth order with C and so passing through the osculating space S_4 to C at P_x. The equations of three independent ones are

$$HR(x_3 - \lambda x_2 - \lambda' x_4) - ERx_7 - M(Hx_8 - Jx_7) = 0,$$

$$HR(x_5 - 2\lambda x_4) - FRx_7 - N(Hx_8 - Jx_7) = 0,$$

$$HR(x_6 - \lambda^2 x_4) - GR x_7 - Q(Hx_8 - Jx_7) = 0.$$

The locus of the osculating spaces S_4 to all curves C that have at P_x the same element E_3 is the space S_5, whose two equations are the last two of the equations of the osculating space S_4. The locus of the osculating spaces S_4 to all curves C that have at P_x the same element E_2 is a hyperplane whose equation is found by eliminating λ'' from these two equations. Since λ'' appears in only Q and N, and since $Q - \lambda N$ is free from λ'' we have for the equation

$$HR(\lambda x_5 - x_6 - \lambda^2 x_4) + [R(G - \lambda F) - J(Q - \lambda N)]x_7 + H(Q - \lambda N)x_8 = 0. \tag{87}$$

To find the hypersurface enveloped by these hyperplanes as λ' varies we must differentiate with respect to λ' and eliminate λ'. For this purpose write

$$Q - \lambda N = T + 6U\lambda' + 3H\lambda'^2,$$

$$R = \bar{R} + 6V\lambda',$$

where

$$U = -4C^3\lambda^3(G - \lambda F) + H(a\lambda^2 + a'\lambda^4 + b + b'\lambda^2 - k\lambda - k'\lambda^3),$$

$$V = H(2\lambda - 3C\lambda^2) - 4C^3 J\lambda^2,$$

and T and $\bar{R}$ are composed of those terms of $Q - \lambda N$ and R which do not involve λ'. H, J, and $G - \lambda F$ do not involve λ'. On differentiating and eliminating λ' we have

$$H(-Jx_7 + Hx_8)\{\bar{R}[H(\lambda x_5 - \lambda^2 x_4 - x_6) + (G - \lambda F)x_7] + T(-Jx_7 + Hx_8)\} - 3\{V[H(\lambda x_5 - \lambda^2 x_4 - x_6) + (G - \lambda F)x_7] + U(-Jx_7 + Hx_8)\}^2 = 0, \tag{88}$$

or

$$-3V^2\left[H(\lambda x_5 - \lambda^2 x_4 - x_6) + (G - \lambda F)x_7\right]^2$$

$$+ (H\bar{R} - 6VU)\left[H(\lambda x_5 - \lambda^2 x_4 - x_6) + (G - \lambda F)x_7\right](-Jx_7 + Hx_8)$$

$$+ (HT - 3U^2)(-Jx_7 + Hx_8)^2 = 0.$$

Equation (88) is the equation of a quadric hypercone V_6^2 in the space S_7, corresponding to the direction $dv - \lambda\, du = 0$, which is the envelope of all the hyperplanes, corresponding to all values of λ', which are the loci of the osculating spaces S_4 to all curves C through P_x that have at P_x the same element E_2. The second form of the equation shows that when the discriminant of the left hand side is zero, that is when

$$H^2\bar{R}^2 - 12H\bar{R}VU + 12HTV^2 = 0,$$

the cone degenerates into a hyperplane counted twice.

Substituting from expansions (74) into equation (88) we find that it is satisfied identically through terms of the third degree in Δu and Δv. The cone therefore has contact of at least the third order with S. Replacing Δu and Δv by du and dv we find that the terms in du and dv of fourth degree are

$$V^2 H^2 (dv - \lambda\, du)^4.$$

By hypothesis $H \neq 0$. When $V \neq 0$, the cone will have contact of exactly the third order with S, and will cut S in a curve with a quadruple point at P_x, the four principal tangents coinciding in the direction $dv - \lambda\, du = 0$. When $V = 0$ the cone will have contact of at least the fourth order. Now the definition of V gives

$$V = 2\lambda\,(1-2C\lambda)(1+gC\lambda),$$

where $g = [1 \pm (-15)^{\frac{1}{2}}]/4$ as defined in connection with equations (67). Thus there are four directions, $dv = 0$, $du - 2C\,dv = 0$, $du + gC\,dv = 0$, such that the cones corresponding to these directions have contact of higher than the third order with S. These directions are the directions of the double families of the four singular triples of one-parameter families of curves on S.[5] We consider each of the cases separately.

(a) $\lambda = 0$. In this case we have the cone corresponding to the direction of the u-curve. On substituting from the expansions (74) into the equation of the cone the fifth degree terms in du and dv are found to be

$$\alpha_2\,dv^4\,(du - C\,dv).$$

We shall see below (Section VI) that, as in the case of the fifth canonical form, $\alpha_2 = 0$ is the condition that the u-curves on S be quasi-asymptotics of the type $\gamma_{2,4}$. Hence if the u-curves on S are not quasi-symptotics of the type $\gamma_{2,4}$, the cone corresponding to the direction of the u-curve will have contact with S of exactly the fourth order. It will cut S in a curve with a quintuple point at P_x, with four of the principal tangents coinciding with the tangent to the u-curve and the fifth lying in the direction $du - C\,dv = 0$. If the u-curves are such quasi-asymptotics, the cone corresponding to the direction of the u-curve will have contact with S of at

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[5] E. P. Lane, loc. cit.

least the fifth order.

(b) $\lambda = 1/2C$. Here we have the cone corresponding to the direction $du - 2\,C\,dv = 0$. On substituting from expansions (74), the fifth degree terms in du and dv, except for factors which are constant or known to be different from zero, are

$$\bar{R}\,(du - 2\,C\,dv)^4\,(du + 4\,C\,dv),$$

where in $\bar{R}$ we have replaced λ by $1/2C$. If this value of $\bar{R}$ does not vanish the cone will have contact with S of exactly the fourth order. It will cut S in a curve with a quintuple point at P_x, with four of the principal tangents lying in the direction $du - 2\,C\,dv = 0$ and the fifth in the direction $du + 4\,C\,dv = 0$. If this value of $\bar{R}$ is zero the cone will have contact with S of at least the fifth order.

(c) $\lambda = -1/gC$. Here we have the cones corresponding to the directions $du + g\,C\,dv = 0$. On substituting from expansions (74) the fifth degree terms in du and dv, except for factors known to be different from zero, are

$$\bar{R}\,(du + g\,C\,dv)^4\,(du - 2\,g\,C\,dv),$$

where in $\bar{R}$ we have replaced λ by $-1/gC$. As in the preceding cases; if $\bar{R}$ is different from zero the cone has contact with S of exactly the fourth order, the directions of the principal tangents at P_x of the curve of intersection being given by the above expression equated to zero; while if $\bar{R}$ is equal to zero the cone will have contact with S of at least the fifth order.

Assuming that $J \neq 0$ and developing the theory of the osculating spaces S_4 to curves C through P_x we get results

similar to those just given. The cones corresponding to the values of λ, given by $H = 0$, which were excluded in the previous case, have contact with S of only the third order. Thus we find no additional directions corresponding to which the cones will have contact of more than third order.

To investigate the osculating spaces to curves through P_x in the direction of the v-curve, we represent such curves by

$$u = u(v), \qquad u'\Big|_{P_x} = 0.$$

The local coordinates of a point near P_x on such a curve and having the parameter value $v + \Delta v$ are:

$$\begin{aligned}
x_1 &= 1 + \tfrac{1}{6}a'\Delta v^3 + \tfrac{1}{24}(\epsilon_3 - 12C^4\alpha_3)\Delta v^4 + \cdots,\\
x_2 &= \tfrac{1}{2}u''\Delta v^2 + \tfrac{1}{6}(l' + u''')\Delta v^3 + \tfrac{1}{24}(\epsilon_6 - 12C^4\alpha_6 + u^{iv})\Delta v^4 + \cdots,\\
x_3 &= \Delta v + \tfrac{1}{6}m'\Delta v^3 + \tfrac{1}{24}(\epsilon_7 - 12C^4\alpha_7)\Delta v^4 + \cdots,\\
x_4 &= \tfrac{1}{6}a'\Delta v^3 + \tfrac{1}{24}(\epsilon_3 - 12C^4\alpha_3 + 3u''^2)\Delta v^4 + \cdots,\\
x_5 &= \tfrac{1}{6}(l' + 3u'')\Delta v^3 + \tfrac{1}{24}(\epsilon_4 - 12C^4\alpha_4 + 4u''')\Delta v^4 + \cdots,\\
x_6 &= \tfrac{1}{2}\Delta v^2 + \tfrac{1}{6}b'\Delta v^3 + \tfrac{1}{24}(\epsilon_5 - 12C^4\alpha_5)\Delta v^4 + \cdots,\\
x_7 &= \tfrac{2}{3}C^3\Delta v^3 + \tfrac{1}{24}(\epsilon_1 - 12C^4\alpha_1)\Delta v^4 + \cdots,\\
x_8 &= -\tfrac{1}{2}C\Delta v^3 + \tfrac{1}{24}(\epsilon_2 - 12C^4\alpha_2 + 6u'')\Delta v^4 + \cdots.
\end{aligned} \tag{89}$$

The theory proceeds in a manner similar to that for the general case. The equation of the cone, which is the envelope of the hyperplanes that are the loci of the osculating spaces S_4 to all curves C that have at P_x the same element E_2, is found to

be

$$3x_7\left[Y\left(x_4+\tfrac{a'}{3C}x_8\right)-Zx_7\right]+\left\{4C^2\left(x_4+\tfrac{a'}{3C}x_8\right)-\tfrac{2a'}{C}x_7\right\}^2=0,$$

where

$$Y=\epsilon_1-12C^4\alpha_1+\tfrac{4}{3}C^2(\epsilon_2-12C^4\alpha_2),$$

$$Z=\epsilon_3-12C^4\alpha_3+\tfrac{a'}{3C}(\epsilon_2-12C^4\alpha_2).$$

None of the properties of this cone here considered distinguish it from the cone corresponding to the general direction at P_x. It has contact of the third order with S at P_x, and the four principal tangents at P_x to the curve of intersection of the cone and S all coincide with the tangent to the v-curve.

3. The Sixth Canonical Form with $K_o = 0$. Consider an integral surface S of equations (41) of the sixth canonical form with $K_o = 0$, which lies in a space S_8, and let P_x be a point on S. Let $dv - \lambda\, du = 0$ be the differential equation of a curve C on S through P_x, and let P_X be a point on C near P_x with parameter value $u + \Delta u$. Then the local coordinates of P_X are:

$$x_1=1+\tfrac{1}{6}(3a\lambda+a'\lambda^3)\Delta u^3+\tfrac{1}{24}(4\beta_8\lambda+6\gamma_8\lambda^2+4\delta_8\lambda^3$$
$$+\epsilon_8\lambda^4+6a\lambda'+6a'\lambda^2\lambda')\Delta u^4+\cdots,$$

$$x_2=\Delta u+\tfrac{1}{6}(3l\lambda+l'\lambda^3)\Delta u^3+\tfrac{1}{24}(4\beta_6\lambda+6\gamma_6\lambda^2+4\delta_6\lambda^3$$
$$+\epsilon_6\lambda^4+6l\lambda'+6l'\lambda^2\lambda')\Delta u^4+\cdots,$$

$$x_3=\lambda\Delta u+\tfrac{1}{2}\lambda'\Delta u^2+\tfrac{1}{6}(3m\lambda+m'\lambda^3+\lambda'')\Delta u^3+\tfrac{1}{24}(4\beta_7\lambda+6\gamma_7\lambda^2$$

$$+ 4\delta_7\lambda^3 + \epsilon_7\lambda^4 + 6m\lambda' + 6m'\lambda^2\lambda' + \lambda''')\Delta u^4 + \cdots,$$

$$x_4 = \tfrac{1}{2}\Delta u^2 + \tfrac{1}{6}(3a\lambda + a'\lambda^3)\Delta u^3 + \tfrac{1}{24}(4\beta_3\lambda + 6\gamma_3\lambda^2 + 4\delta_3\lambda^3$$

$$+ \epsilon_3\lambda^4 + 6a\lambda' + 6a'\lambda^2\lambda')\Delta u^4 + \cdots,$$

(90)

$$x_5 = \lambda\Delta u^2 + \tfrac{1}{6}(3k\lambda + k'\lambda^3 + 3\lambda')\Delta u^3 + \tfrac{1}{24}(4\beta_4\lambda + 6\gamma_4\lambda^2$$

$$+ 4\delta_4\lambda^3 + \epsilon_4\lambda^4 + 6k\lambda' + 6k'\lambda^2\lambda' + 4\lambda'')\Delta u^4 + \cdots,$$

$$x_6 = \tfrac{1}{2}\lambda^2\Delta u^2 + \tfrac{1}{6}(3k\lambda + k'\lambda^3 + 3\lambda\lambda')\Delta u^3 + \tfrac{1}{24}(4\beta_5\lambda + 6\gamma_5\lambda^2$$

$$+ 4\delta_5\lambda^3 + \epsilon_5\lambda^4 + 6k\lambda' + 6k'\lambda^2\lambda' + 3\lambda'^2 + 4\lambda\lambda'')\Delta u^4 + \cdots,$$

$$x_7 = \tfrac{1}{6}(1 + 4C^3\lambda^3)\Delta u^3 + \tfrac{1}{24}(4\beta_1\lambda + 6\gamma_1\lambda^2 + 4\delta_1\lambda^3$$

$$+ \epsilon_1\lambda^4 + 24C^3\lambda^2\lambda')\Delta u^4 + \cdots,$$

$$x_8 = \tfrac{1}{2}(\lambda^2 - C\lambda^3)\Delta u^3 + \tfrac{1}{24}(4\beta_2\lambda + 6\gamma_2\lambda^2 + 4\delta_2\lambda^3$$

$$+ \epsilon_2\lambda^4 + 12\lambda\lambda' - 18C\lambda^2\lambda')\Delta u^4 + \cdots,$$

$$x_9 = \tfrac{1}{24}(1 + 16C^3\lambda^3 - 12C^4\lambda^4)\Delta u^4 + \cdots.$$

All of the osculating spaces of one, two, and three dimensions, and all of the loci determined by them, are exactly the same here as in the case $K_0 \neq 0$. All of these loci lie in the hyperplane $x_9 = 0$, and regarding this as their ambient space their equations are identical with those obtained in the preceding case. Regarding the space S_8, in which S lies, as their ambient space we find their equations by adding to the sets of equations found in the preceding case the equation $x_9 = 0$.

The equation of the ∞^4 hyperplanes through the osculating space S_3 to C at P_x may be written in either of the two forms,

$$\xi_3[H(x_3 - \lambda x_2 - \lambda' x_4) - E x_7] + \xi_5[H(x_5 - 2\lambda x_4) - F x_7]$$
$$+ \xi_6[H(x_6 - \lambda^2 x_4) - G x_7] + \xi_8[H x_8 - J x_7] + \xi_9 H x_9 = 0,$$

or

$$\xi_3[J(x_3 - \lambda x_2 - \lambda' x_4) - E x_8] + \xi_5[J(x_5 - 2\lambda x_4) - F x_8]$$
$$+ \xi_6[J(x_6 - \lambda^2 x_4) - G x_8] + \xi_7[J x_7 - H x_8] + \xi_9 J x_9 = 0,$$

where E, F, G, H, J have the same values as in the case $K_0 \neq 0$. The first of these is valid for all values of λ such that $H \neq 0$, and the second for all values of λ such that $J \neq 0$. We consider the first of these equations. In order to find the hyperplanes through the osculating space S_4 to C at P_x we must require that this equation be satisfied identically by the expansions (90) through terms in Δu^4. This imposes the condition on the ξ's,

$$M\xi_3 + N\xi_5 + Q\xi_6 + R\xi_8 + X\xi_9 = 0,$$

where the values of M, N, Q, R are found from the corresponding values in the case $K_0 \neq 0$ by suppressing all terms involving the α's, and where

$$X = H(1 + 16C^3\lambda^3 - 12C^4\lambda^4).$$

If λ is one of the four directions that makes $X = 0$ then the analysis will proceed in the same manner as in the case $K_0 \neq 0$. All the figures there considered will be obtained here and will lie in the hyperplane $x_9 = 0$. Their equations in this

case are found by adjoining to their equations in the preceding case the equation $x_9 = 0$, and by making the change in the values of M, N, Q, R. We note that these four directions are the directions of the principal tangents at P_x to the curve of intersection of S and the hyperplane $x_9 = 0$. (See Section IV, 3).

Suppose now that $X \neq 0$. Then there will be ∞^3 hyperplanes with contact of the fourth order with C at P_x and so passing through the osculating space S_4 to C at that point. The equations of four independent ones are

$$HX(x_3 - \lambda x_2 - \lambda' x_4) - EX x_7 - HM x_9 = 0,$$

$$HX(x_5 - 2\lambda x_4) - FX x_7 - HN x_9 = 0,$$

$$HX(x_6 - \lambda^2 x_4) - GX x_7 - HQ x_9 = 0,$$

$$HX x_8 - JX x_7 - HR x_9 = 0.$$

The locus of the osculating spaces S_4 to all curves C that have at P_x the same element E_3 is a space S_5, whose three equations are obtained by omitting the first of the four equations of the osculating space S_4. The equations of the locus of the osculating spaces S_4 to all curves C that have at P_x the same element E_2 is found by eliminating λ' from the three equations of the space S_5 just considered. The locus is a space S_6 with equations

$$HX(\lambda x_5 - x_6 - \lambda^2 x_4) + X(G - \lambda F)x_7 + H(Q - \lambda N)x_9 = 0,$$

$$HX x_8 - JX x_7 - HR x_9 = 0.$$

In this case we may eliminate λ' from these two equations and so obtain the equation of the locus of the osculat-

ing spaces S_4 to all curves C through P_x that have there the same element E_1. For this purpose we write

$$Q - \lambda N = T + 6 U \lambda' + 3 H \lambda'^2,$$

$$R = \bar{R} + 6 V \lambda' ,$$

where U and V have the same values as in the case $K_o \neq 0$, and T and $\bar{R}$ are composed of those terms of $Q - \lambda N$ and R which do not involve λ'. H, J, $G - \lambda F$, and X do not involve λ'. On eliminating we have

$$\begin{aligned} (91) \qquad & 12 V^2 x_9 \left[X H (\lambda x_5 - x_6 - \lambda^2 x_4) + X (G - \lambda F) x_7 + H T x_9 \right] \\ & + 12 U V x_9 \left[X (H x_8 - J x_7) - H \bar{R} x_9 \right] \\ & + \left[X (H x_8 - J x_7) - H \bar{R} x_9 \right]^2 = 0. \end{aligned}$$

The locus is thus seen to be a quadric hypercone, a V_7^2, lying in the space S_8 which is the ambient space of S. On substituting from expansions (77) into equation (91) we find that the equation is satisfied identically by the expansions through fifth degree terms in Δu and Δv. The cone therefore has contact with S of the fifth order. There will be no directions, except those for which X = 0 and H = 0 which we have excluded, such that the cone corresponding to these directions will have contact with S of higher than the fifth order.

The cone may be characterized geometrically in the following way. The osculating space S_4 to C at P_x is determined by the five points with coordinates x, x', x'', x''', x^{IV}. Write

$$x' = x_u + x_v \lambda ,$$

$$x'' = P_1 + x_v \lambda' ,$$

$$x''' = P_2 + 3 P_3 \lambda' + x_v \lambda'' ,$$

$$x'^{v} = P_4 + 6\,P_5\lambda' + 3\,x_{vv}\lambda'^2 + 4\,P_3\lambda'' + x_v\,\lambda''' ,$$

where

$$P_1 = x_{uu} + 2\,x_{uv}\lambda + x_{vv}\lambda^2 ,$$

$$P_2 = x_{uuu} + 3\,x_{uuv}\lambda + 3\,x_{uvv}\lambda^2 + x_{vvv}\lambda^3 ,$$

$$P_3 = x_{uv} + x_{vv}\lambda ,$$

$$P_4 = x_{uuuu} + 4\,x_{uuuv}\lambda + 6\,x_{uuvv}\lambda^2 + 4\,x_{uvvv}\lambda^3 + x_{vvvv}\lambda^4 ,$$

$$P_5 = x_{uuv} + 2\,x_{uvv}\lambda + x_{vvv}\lambda^2 .$$

To find the locus of the osculating spaces S_4 to all curves C that have at P_x the same element E_3 we hold λ, λ', λ'' fast and let λ''' vary. The point x'^{v} then moves along the line joining $P_4 + 6\,P_5\lambda' + 3\,x_{vv}\lambda'^2 + 4\,P_3\lambda''$ to x_v. Hence the osculating spaces S_4 form a pencil with the osculating space S_3 as axis, and the pencil fills a space S_5 which is the desired locus. This space S_5 is determined by the six independent points x, x_u, x_v, P_1, $P_2 + 3\,P_3\lambda'$, $P_4 + 6\,P_5\lambda' + 3\,x_{vv}\lambda'^2 + 4\,P_3\lambda''$. To find the locus of the osculating spaces S_4 to all curves C that have at P_x the same element E_2 we hold λ, λ' fast and let λ'' vary. The point $P_4 + 6\,P_5\lambda' + 3\,x_{vv}\lambda'^2 + 4\,P_3\lambda''$ will move along the line joining $P_4 + 6\,P_5\lambda' + 3\,x_{vv}\lambda'^2$ to P_3. The spaces S_5 just considered will form a pencil, whose axis is the space S_4 determined by the points x, x_u, x_v, P_1, $P_2 + 3\,P_3\lambda'$, and which fills a space S_6 which is the desired locus. This space S_6 is determined by the seven independent points x, x_u, x_v, P_1, P_2, P_3, and $P_4 + 6\,P_5\lambda' + 3\,x_{vv}\lambda'^2$. To find the locus of the osculating spaces S_4 to all curves C that have at P_x the same tangent line we hold λ fast and let λ' vary. Under

these conditions the first six of the seven points just considered remain fixed, while the seventh describes a conic in the plane determined by x_{vv}, P_4, P_5. Hence the quadric hypersurface defined by equation (91) is a quadric hypercone whose vertex is a space S_5 determined by the points x, x_u, x_v, P_1, P_2, P_3, and whose elements are spaces S_6 joining this vertex to the point $P_4 + 6\,P_5\lambda' + 3\,x_{vv}\lambda'^2$ which as λ' varies describes a conic in the plane of x_{vv}, P_4, P_5. It is evident from their definitions that the nine points, the six determining the vertex and the three determining the plane of the conic, are independent; and so the cone is immersed in the space S_8 which is the ambient space of S.

To investigate the osculating spaces to curves C through P_x in the direction of the v-curve, we represent such curves by

$$u = u(v), \qquad u'\Big|_{P_x} = 0.$$

If P_X is a point on C near P_x with parameter value $v + \Delta v$, its local coordinates expressed as power series in Δv are:

$$\begin{aligned}
X_1 &= 1 + \tfrac{1}{6}\alpha'\Delta v^3 + \tfrac{1}{24}\epsilon_8\Delta v^4 + \cdots,\\
X_2 &= \tfrac{1}{2}u''\Delta v^2 + \tfrac{1}{6}(\beta' + u''')\Delta v^3 + \tfrac{1}{24}(\epsilon_6 + u^{iv})\Delta v^4 + \cdots,\\
X_3 &= \Delta v + \tfrac{1}{6}m'\Delta v^3 + \tfrac{1}{24}\epsilon_7\Delta v^4 + \cdots,\\
X_4 &= \tfrac{1}{6}a'\Delta v^3 + \tfrac{1}{24}(\epsilon_3 + 3u''^2)\Delta v^4 + \cdots,\\
X_5 &= \tfrac{1}{6}(b' + 3u'')\Delta v^3 + \tfrac{1}{24}(\epsilon_4 + 4u''')\Delta v^4 + \cdots,\\
X_6 &= \tfrac{1}{2}\Delta v^2 + \tfrac{1}{6}b'\Delta v^3 + \tfrac{1}{24}\epsilon_5\Delta v^4 + \cdots,
\end{aligned} \tag{92}$$

$$x_7 = \frac{2}{3} C^3 \Delta v^3 + \frac{1}{24} \epsilon_1 \Delta v^4 + \cdots,$$

$$x_8 = -\frac{1}{2} C \Delta v^3 + \frac{1}{24} (\epsilon_2 + 6 a'') \Delta v^4 + \cdots,$$

$$x_9 = -\frac{1}{2} C^4 \Delta v^4 + \cdots.$$

The theory here proceeds as in the general case, none of the figures possessing properties different from those corresponding to a general curve. The equation of the cone which is the locus of the osculating spaces S_4 to all curves C through P_x in the direction of the v-curve is

$$64 C^4 x_9 \left[12 C^4 x_4 + 4 C^3 a' x_8 + (\epsilon_3 + \tfrac{a'}{3C} \epsilon_2) x_9 \right]$$
$$- 16 C a' x_9 \left[12 C^4 x_7 + 16 C^6 x_8 + (\epsilon_1 + \tfrac{4}{3} C^2 \epsilon_2) x_9 \right]$$
$$+ 3 \left[12 C^4 x_7 + 16 C^6 x_8 + (\epsilon_1 + \tfrac{4}{3} C^2 \epsilon_2) x_9 \right]^2 = 0.$$

It has contact with S of the fifth order as does the corresponding cone in the general case.

SECTION VI

QUASI-ASYMPTOTICS

On a surface S in a space S_n ($n \geq 4$) there are in general no asymptotic curves. In their place Bompiani has introduced quasi-asymptotics which may be defined in the following way.[1] A quasi-asymptotic curve C of type $\gamma_{p,q}$ ($p < q$) is a curve such that the osculating space S(p,0) to the surface S and the osculating space S_q to C at every point P of C lie in a space of smaller dimensionality than for the general curve on S. In this section we shall develop a method which will determine in general the number and types of quasi-asymptotics on a surface in a given space S_n, and shall apply this theory to the particular surfaces we are studying.

1. General Theory. Consider a space S_n of a definite number of dimensions and in it a surface S which is the integral surface of a system of one or more differential equations. On S consider a curve C which is one of the family with differential equation $dv - \lambda\, du = 0$. Then the osculating space S_q to C at a point P_x on C is determined by the q + 1 points, x, x', x'', ---, $x^{(q)}$. The osculating space S(p,0) to S at P_x is determined by the points x, x_u, x_v, x_{uu}, x_{uv}, x_{vv}, --- through the p-th derivative points of x, and is a space S_m where $m \leq$

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[1] E. Bompiani, Sopra alcune estensione dei teoremi di Meusnier e di Eulero. Atti della R. Acc. delle Scienze di Torino, vol. 48 (1913), pp. 393-410.

$p(p+3)/2$, the inequality holding when some of the derivatives of x are connected by the system of differential equations defining the surface S. It is at once evident that the points $x, x', x'', \cdots, x^{(p)}$, which are part of those defining the osculating space S_q to C will lie in the osculating space S(p,0) to S. If the value of p is such that $m = n$, then the osculating spaces S_q to all curves C on S and for all values of q will lie in the space S(p,0), and there will be no curve C on S such that the two osculating spaces S_q and S(p,0) will lie in a space of fewer dimensions than for a general curve. Hence in seeking quasi-asymptotics we need consider only those values of p which make $m < n$.

Denote by $r+1$ the total number of independent points in the two osculating spaces S(p,0) at a point of S and S_q to a general curve C on S at the point. Then $r = m + q - p$. If now p and q are such that $r = n$ then the surface S will certain quasi-asymptotics of the type $\gamma_{p,q}$. To show this we note that under these conditions the ambient space of the two osculating spaces S(p,0) and S_q is the space S_n, and that in order that C be a quasi-asymptotic this ambient space must reduce to a hyperplane or a space of fewer dimensions. Analytically this means that the $n+1$ by $n+1$ matrix of the coordinates of the $n+1$ points must be of rank n, that is, its determinant must vanish. Since the elements of the determinant are functions of u, v, λ, and the derivatives of λ, the vanishing of the determinant gives a partial differential equation in λ, and every λ which is a solution of this equation

makes C a quasi-asymptotic of type $\gamma_{p,q}$ on S. Since $\lambda = dv/du$, the differential equation is a total differential equation in u and v, and is the differential equation of these quasi-asymptotics on S.

If p and q have such values that $r < n$ then in general there will be no quasi-asymptotics of type $\gamma_{p,q}$ on S. Under these conditions the ambient space of the two osculating spaces $S(p,0)$ and S_q is a space S_r for the general curve C on S, and in order that C be a quasi-asymptotic it is necessary that this space reduce to a space of not more than r - 1 dimensions. This means that the r + 1 by n + 1 matrix of the coefficients of the r + 1 points must be of rank r, and hence that the determinants of all r + 1 by r + 1 minors of the matrix must vanish. As before, the vanishing of these determinants yields total differential equations in u and v, and since two or more such equations will not in general have a common solution, there will in general be no quasi-asymptotics of type $\gamma_{p,q}$ on S. A similar argument shows that when p and q are such that $r > n$ there will not in general be quasi-asymptotics of type $\gamma_{p,q}$ on S.

Summarizing these results we have the following theorem: In order that a surface S in a space S_n have on it quasi-asymptotics of type $\gamma_{p,q}$ it is necessary that p have such a value that the osculating space $S(p,0)$ is of lower dimensionality than the space S_n; and in general the surface S will contain such quasi-asymptotics if and only if the number of independent points in the osculating space $S(p,0)$ to S and in

the osculating space S_q to a general curve C on S is equal to n + 1. From this it follows that for a given surface S and a given value of p such that the osculating space S(p,0) is of lower dimensionality than the space S_n, there will be in general a unique value of q such that the surface S will contain quasi-asymptotics of the type $y_{p,q}$. From the definition of r and the condition that n = r, we have for this value of q

$$q = n + p - m.$$

The following table gives the types of quasi-asymptotics corresponding to certain values of p and m.

p	Form of Differential Equations defining S	m	q	Types of Quasi-Asymp.
1	any system whatever	2	n-1	$y_{1,n-1}$
2	no second order equations	5	n-3	$y_{2,n-3}$
	one second order equation	4	n-2	$y_{2,n-2}$
	two second order equations	3	---	---------
3	no second order equations no third order equations	9	n-6	$y_{3,n-6}$
	no second order equations one third order equation	8	n-5	$y_{3,n-5}$
	no second order equations two third order equations	7	n-4	$y_{3,n-4}$

In the case p = 2, m = 3, we also have n = 3 and so there are no quasi-asymptotics.

2. The Fifth Canonical Form with $a' \neq 0$. Since an integral surface of a pair of differential equations in the

fifth canonical form with $a' \neq 0$ lies in a space S_7, we find from the preceding theory that such a surface will contain quasi-asymptotics of just two types, $y_{1,6}$ and $y_{2,4}$. In order to find the differential equations of these quasi-asymptotics it would be necessary to compute the values of x'', x''', x^{IV}, x^{V}, x^{VI}. From the theory of Section II we know that when the integrability conditions (36) are satisfied, all of these derivatives can be computed uniquely as linear combinations of x, x_u, x_v, x_{uu}, x_{uv}, x_{vv}, x_{uuu}, x_{vvv}. We indicate the results as follows:

$$x'' = x_{uu} + 2\lambda x_{uv} + \lambda^2 x_{vv} + \lambda' x_v,$$

$$x''' = x_{uuu} + (3b\lambda^2 + \lambda^3)x_{vvv} + \ell_3 x_{uu} + \ell_4 x_{uv} + \ell_5 x_{vv} + \ell_6 x_u + \ell_7 x_v + \ell_8 x,$$

$$x^{IV} = f_1 x_{uuu} + f_2 x_{vvv} + f_3 x_{uu} + f_4 x_{uv} + f_5 x_{vv} + f_6 x_u + f_7 x_v + f_8 x,$$

$$x^{V} = g_1 x_{uuu} + g_2 x_{vvv} + g_3 x_{uu} + g_4 x_{uv} + g_5 x_{vv} + g_6 x_u + g_7 x_v + g_8 x,$$

$$x^{VI} = h_1 x_{uuu} + h_2 x_{vvv} + h_3 x_{uu} + h_4 x_{uv} + h_5 x_{vv} + h_6 x_u + h_7 x_v + h_8 x,$$

where

$$\ell_3 = 3a\lambda + 3a'\lambda^2; \quad \ell_4 = 3b\lambda + 3b'\lambda^2 + 3\lambda',$$

$$\ell_5 = 3b\lambda + 3b'\lambda^2 + 3\lambda\lambda', \quad \ell_6 = 3\ell\lambda + 3\ell'\lambda^2,$$

$$\ell_7 = 3m\lambda + 3m'\lambda^2 + \lambda'', \quad \ell_8 = 3d\lambda + 3d'\lambda^2,$$

$$f_1 = \alpha_1 + 4\beta_1\lambda + 4\delta_1\lambda^3 + \epsilon_1\lambda^4,$$

$$f_2 = \alpha_2 + 4\beta_2\lambda + 6\gamma_2\lambda^2 + 4\delta_2\lambda^3 + \epsilon_2\lambda^4 + 12D\lambda\lambda' + 6\lambda^2\lambda' + 3\lambda'^2,$$

and where the other coefficients can be computed by successive differentiation and substitution.

The differential equation of the quasi-asymptotics of type $\gamma_{1,6}$ is found from

$$(x, x_u, x_v, x'', x''', x^{iv}, x^v, x^{vi}) = 0,$$

where the parentheses are used to indicate the determinant of the coordinates of these points. This gives:

(93) $$(x, x_u, x_v, x_{uu} + 2\lambda x_{uv} + \lambda^2 x_{vv},$$

$$x_{uuu} + (3D\lambda^2 + \lambda^3)x_{vvv} + \ell_3 x_{uu} + \ell_4 x_{uv} + \ell_5 x_{vv},$$

$$f_1 x_{uuu} + f_2 x_{vvv} + f_3 x_{uu} + f_4 x_{uv} + f_5 x_{vv},$$

$$g_1 x_{uuu} + g_2 x_{vvv} + g_3 x_{uu} + g_4 x_{uv} + g_5 x_{vv},$$

$$\ell_1 x_{uuu} + \ell_2 x_{vvv} + \ell_3 x_{uu} + \ell_4 x_{uv} + \ell_5 x_{vv}) = 0,$$

which may be broken up into a sum of determinants. The determinant

$$W = (x, x_u, x_v, x_{uu}, x_{uv}, x_{vv}, x_{uuu}, x_{vvv}),$$

which is not zero, appears in each term, and on removing this we have the differential equation of the quasi-asymptotics of type $\gamma_{1,6}$. This equation is of fifth order and so represents a five-fold infinity of quasi-asymptotics of type $\gamma_{1,6}$ on S.

The differential equation of the quasi-asymptotics of

type $y_{2,4}$ is found by equating to zero the determinant

$$(x,\ x_u,\ x_v,\ x_{uu},\ x_{uv},\ x_{vv},\ x''',\ x^{IV}),$$

which gives

$$(x,\ x_u,\ x_v,\ x_{uu},\ x_{uv},\ x_{vv},\ x_{uuu} + (3D\lambda^2 + \lambda^3)\ x_{vvv},\ f_1 x_{uuu} + f_2 x_{vvv}) = 0.$$

Removing the factor W we have for the desired equation

$$f_2 - (3D\lambda^2 + \lambda^3) f_1 = 0. \tag{94}$$

This equation, regarded as a total differential equation in u and v, is of second order and second degree, and so represents a double infinity of quasi-asymptotics of type $y_{2,4}$ on S. When $\alpha_2 = 0$, $\lambda = 0$ is a solution of equation (94), and so $\alpha_2 = 0$ is the condition that the u-curves on S be quasi-asymptotics of this type.

By choosing the differential equation of C in the form that we have used, we have excluded from our consideration the v-curves on S. By a separate argument it can be shown that the v-curves can be quasi-asymptotics of the type $y_{1,6}$ only in case the coefficients of the system (26) satisfy a certain condition, and that they can never be quasi-asymptotics of type $y_{2,4}$.

3. The Fifth Canonical Form with $a' \neq 0$. In this case all derivatives of x with respect to u alone are linearly independent, and an integral surface S lies in a space S_n with $n \geq 7$. Since the dimensionality of S_n is not a definite number the general theory developed in the first part of this section is not applicable. Representing a curve C on S by

$dv - \lambda\, du = 0$ we note that the expression for $x^{(j)}$ involves $\partial^j x / \partial u^j$ with coefficient unity. Hence for no value of λ will the osculating spaces S_q and $S(p,0)$ lie in a space of fewer dimensions than for the general curve. Thus no curve represented by $dv - \lambda\, du = 0$ can be a quasi-asymptotic on S.

Considering the v-curves on S, we find that the osculating space S_q to a v-curve is determined by the points x, x_v, x_{vv}, ---, $\partial^j x / \partial v^j$. The points x, x_v, x_{vv}, x_{vvv} are independent points. But, since $\delta_i = 0$ and $\epsilon_i = 0$ when $a^i = 0$, the value of $\partial^j x / \partial v^j$ for $q > 3$ is expressible as a linear combination of the points x, x_u, x_v, x_{uu}, x_{uv}, x_{vv}, x_{vvv}. In general the points x_{vvvv}, x_{vvvvv}, x_{vvvvvv}, being different linear combinations of these seven points, are also independent, so that the v-curves have osculating spaces of one to six dimensions, but a v-curve can have no osculating space of dimensionality greater than six. That is, each v-curve is immersed in a space S_6. Thus the only values of p that need be considered in investigating whether or not the v-curves are quasi-asymptotics are those of five or less. We find that the v-curves may be regarded as quasi-asymptotics of all of the following types:

$y_{1,6}$,

$y_{2,q}$, $(q = 4,5,6)$,

$y_{3,q}$, $(q = 4,5,6)$,

$y_{4,q}$, $(q = 5,6)$,

$y_{5,6}$.

4. The Sixth Canonical Form with $K_0 \neq 0$. In this case an integral surface S lies in a space S_7, and the general theory shows that S will contain quasi-asymptotics of the types $y_{1,4}$ and $y_{2,4}$. To compute their differential equations we shall need the following derivatives of x:

$$
\begin{aligned}
x''' &= \ell_1 x_{uuu} + \ell_2 x_{uvv} + \ell_3 x_{uu} + \ell_4 x_{uv} + \ell_5 x_{vv} \\
&\quad + \ell_6 x_u + \ell_7 x_v + \ell_8 x, \\
x'^{v} &= f_1 x_{uuu} + f_2 x_{uvv} + f_3 x_{uu} + f_4 x_{uv} + f_5 x_{vv} \\
&\quad + f_6 x_u + f_7 x_v + f_8 x, \\
x^{v} &= g_1 x_{uuu} + g_2 x_{uvv} + g_3 x_{uu} + g_4 x_{uv} + g_5 x_{vv} \\
&\quad + g_6 x_u + g_7 x_v + g_8 x, \\
x^{vi} &= h_1 x_{uuu} + h_2 x_{uvv} + h_3 x_{uu} + h_4 x_{uv} + h_5 x_{vv} \\
&\quad + h_6 x_u + h_7 x_v + h_8 x,
\end{aligned}
$$

where

$$
\begin{aligned}
&\ell_1 = 1 + 4C^3\lambda^3, \qquad \ell_2 = 3\lambda^2 - 3C\lambda^3, \\
&\ell_3 = 3a\lambda + a'\lambda^3, \qquad \ell_4 = 3h\lambda + h'\lambda^3 + 3\lambda', \\
&\ell_5 = 3b\lambda + b'\lambda^3 + 3\lambda\lambda', \quad \ell_6 = 3l\lambda + l'\lambda^3, \\
&\ell_7 = 3m\lambda + m'\lambda^3 + \lambda'', \quad \ell_8 = 3d\lambda + d'\lambda^3, \\
&f_1 = \alpha_1 + 4\beta_1\lambda + 6\gamma_1\lambda^2 + 4(\delta_1 + 4C^3\alpha_1)\lambda^3 \\
&\quad + (\epsilon_1 - 12C^4\alpha_1)\lambda^4 + 24C^3\lambda^2\lambda' + 12C^3\lambda'^2, \\
&f_2 = \alpha_2 + 4\beta_2\lambda + 6\gamma_2\lambda^2 + 4(\delta_2 + 4C^3\alpha_2)\lambda^3 \\
&\quad + (\epsilon_2 - 12C^4\alpha_2)\lambda^4 + 12\lambda\lambda' - 18\lambda^2\lambda' - 9C\lambda'^2,
\end{aligned}
$$

and where the other coefficients can be calculated by successive differentiation and substitution.

The differential equation of the quasi-asymptotics of type $\gamma_{1,6}$ is found from the determinant,

$$(x,\ x_u,\ x_v,\ x'',\ x''',\ x'^{v},\ x^{v},\ x^{v'}).$$

Equating this to zero we have

$$(95)\quad \begin{aligned}&(x,\ x_u,\ x_v,\ x_{uu}+2\lambda x_{uv}+\lambda^2 x_{vv},\\ &e_1 x_{uuu}+e_2 x_{uvv}+e_3 x_{uu}+e_4 x_{uv}+e_5 x_{vv},\\ &f_1 x_{uuu}+f_2 x_{uvv}+f_3 x_{uu}+f_4 x_{uv}+f_5 x_{vv},\\ &g_1 x_{uuu}+g_2 x_{uvv}+g_3 x_{uu}+g_4 x_{uv}+g_5 x_{vv},\\ &h_1 x_{uuu}+h_2 x_{uvv}+h_3 x_{uu}+h_4 x_{uv}+h_5 x_{vv})=0,\end{aligned}$$

and on removing from this equation the determinant

$$W=(x,\ x_u,\ x_v,\ x_{uu},\ x_{uv},\ x_{vv},\ x_{uuu},\ x_{uvv})\neq 0,$$

which appears as a factor, we have left the desired equation. This equation is of fifth order and represents a five-fold infinity of quasi-asymptotics of type $\gamma_{1,6}$ on S.

The differential equation of the quasi-asymptotics of type $\gamma_{2,4}$ is found by equating to zero the determinant,

$$(x,\ x_u,\ x_v,\ x_{uu},\ x_{uv},\ x_{vv},\ x''',\ x'^{v}).$$

Substituting for x''' and x'^{v} their values, and removing the factor W, we have for the desired equation

$$(96)\quad e_1 f_2 - e_2 f_1 = 0.$$

This is a total differential equation in u and v, of second order and second degree, and so represents a double infinity of quasi-asymptotics of type $\gamma_{2,4}$ on S. The only term not involving λ or its derivatives is α_2. Hence $\alpha_2 = 0$ is the condition that $\lambda = 0$ be a solution of equation (96), and that the u-curves on S be quasi-asymptotics of type $\gamma_{2,4}$.

The v-curves on S will not be quasi-asymptotics of either of the two types unless the coefficients of the pair of differential equations (41) satisfy certain conditions.

5. The Sixth Canonical Form with $K_0 = 0$. In this case an integral surface S lies in a space S_8, and we find from the general theory that it will contain quasi-asymptotics of the three types $y_{1,7}$, $y_{2,5}$, and $y_{3,4}$. To compute their equations we shall need the following derivatives of x:

$$x''' = e_1 x_{uuu} + e_2 x_{uvv} + e_3 x_{uu} + e_4 x_{uv} + e_5 x_{vv} + e_6 x_u + e_7 x_v + e_8 x,$$

$$x^{iv} = f_0 x_{uuuu} + f_1 x_{uuu} + f_2 x_{uvv} + f_3 x_{uu} + f_4 x_{uv} + f_5 x_{vv} + f_6 x_u + f_7 x_v + f_8 x,$$

$$x^{v} = g_0 x_{uuuu} + g_1 x_{uuu} + g_2 x_{uvv} + g_3 x_{uu} + g_4 x_{uv} + g_5 x_{vv} + g_6 x_u + g_7 x_v + g_8 x,$$

$$x^{vi} = h_0 x_{uuuu} + h_1 x_{uuu} + h_2 x_{uvv} + h_3 x_{uu} + h_4 x_{uv} + h_5 x_{vv} + h_6 x_u + h_7 x_v + h_8 x,$$

$$x^{vii} = j_0 x_{uuuu} + j_1 x_{uuu} + j_2 x_{uvv} + j_3 x_{uu} + j_4 x_{uv} + j_5 x_{vv} + j_6 x_u + j_7 x_v + j_8 x,$$

where the e's have the same values as in the case $K_0 \neq 0$, where

$$f_0 = 1 + 16 C^3 \lambda^3 - 12 C^4 \lambda^4,$$

$$f_1 = 4\beta_1 \lambda + 6\gamma_1 \lambda^2 + 4\delta_1 \lambda^3 + \epsilon_1 \lambda^4 + 24 C^3 \lambda^2 \lambda' + 12 C^3 \lambda'^2,$$

$$f_2 = 4\beta_2\lambda + 6\gamma_2\lambda^2 + 4\delta_2\lambda^3 + \epsilon_2\lambda^4 + 12\lambda\lambda' - 18C\lambda^2\lambda' - 9C\lambda'^2,$$

$$g_0 = f_0' + f_0 A_0 + f_0 B_0\lambda + f_1 + 4C^3 f_2\lambda,$$

$$g_1 = f_1' + f_0 A_1 + f_0 B_1\lambda + f_1\beta_1\lambda + f_2\gamma_1 + f_2\delta_1\lambda + f_3 + 4C^3 f_5\lambda,$$

$$g_2 = f_2' + f_0 A_2 + f_0 B_2\lambda + f_1\beta_2\lambda + f_2\gamma_2 + f_2\delta_2\lambda + f_4\lambda + f_5 - 3Cf_5\lambda,$$

the accents on the f's denoting total derivatives with respect to u, and the A's and B's being defined by equations (45), and where the other coefficients may be computed by successive differentiation and substitution.

The differential equation of the quasi-asymptotics of type $\gamma_{1,7}$ is found from the determinant,

$$(x,\ x_u,\ x_v,\ x'',\ x''',\ x^{IV},\ x^{V},\ x^{VI},\ x^{VII}).$$

Substituting for the derivatives their values, and equating the determinant to zero, we have

(97) $$\begin{vmatrix} x, & x_u, & x_v, & x_{uu} + 2\lambda x_{uv} + \lambda^2 x_{vv}, \\ \ell_1 x_{uuu} + \ell_2 x_{uvv} + \ell_3 x_{uu} + \ell_4 x_{uv} + \ell_5 x_{vv}, \\ f_0 x_{uuuu} + f_1 x_{uuu} + f_2 x_{uvv} + f_3 x_{uu} + f_4 x_{uv} + f_5 x_{vv}, \\ g_0 x_{uuuu} + g_1 x_{uuu} + g_2 x_{uvv} + g_3 x_{uu} + g_4 x_{uv} + g_5 x_{vv}, \\ h_0 x_{uuuu} + h_1 x_{uuu} + h_2 x_{uvv} + h_3 x_{uu} + h_4 x_{uv} + h_5 x_{vv}, \\ j_0 x_{uuuu} + j_1 x_{uuu} + j_2 x_{uvv} + j_3 x_{uu} + j_4 x_{uv} + j_5 x_{vv} \end{vmatrix} = 0.$$

The left hand side of this equation contains as a factor the determinant,

$$W = (x,\ x_u,\ x_v,\ x_{uu},\ x_{uv},\ x_{vv},\ x_{uuu},\ x_{uvv},\ x_{uuuu}) \neq 0,$$

and on removing this factor we have left the desired equation. This equation is of sixth order and so represents a six-fold infinity of quasi-asymptotics of type $\gamma_{1,7}$ on S.

The differential equation of the quasi-asymptotics of type $\gamma_{2,5}$ is found by equating to zero the determinant,

$$(x,\ x_u,\ x_v,\ x_{uu},\ x_{uv},\ x_{vv},\ x''',\ x'^{v},\ x^{v}).$$

On substituting for the derivatives their values, and dividing by W, we have for the desired equation

$$e_1 f_2 g_0 - e_1 f_0 g_2 - e_2 f_1 g_0 + e_2 f_0 g_1 = 0. \tag{98}$$

This is a third order total differential equation in u and v, and so represents a triple infinity of quasi-asymptotics of type $\gamma_{2,5}$ on S. $A_2 = 0$ is the condition that the u-curves be quasi-asymptotics of this type.

The differential equation of the quasi-asymptotics of type $\gamma_{3,4}$ is found from the determinant,

$$(x,\ x_u,\ x_v,\ x_{uu},\ x_{uv},\ x_{vv},\ x_{uuu},\ x_{uvv},\ x'^{v}),$$

to be

$$1 + 16\, c^3\, \lambda^3 - 12\, c^4\, \lambda^4 = 0. \tag{99}$$

Hence through each point on S there are four quasi-asymptotics of type $\gamma_{3,4}$, and their directions coincide with the directions of the principal tangents to the curve of intersection of S and the unique hyperplane, $x_9 = 0$, which has with S con-

tact of the third order at the point. (See Section IV, 3).

The v-curves on S will not be quasi-asymptotics of types $\gamma_{1,7}$ and $\gamma_{2,5}$ unless the coefficients of the differential equations (41) satisfy certain conditions, and will never be quasi-asymptotics of type $\gamma_{3,4}$

SECTION VII

TABULAR SUMMARY

The following table presents in condensed form the principal results obtained in the preceding work.

		Ambient space
I	Fifth Canonical Form: $a' \neq 0$.	S_7
II	Sixth Canonical Form: $K_o \neq 0$.	S_7
III	Sixth Canonical Form: $K_o = 0$.	S_8

	Transformations that preserve forms of equations
I	Any transformation of dependent variable Two transformations of independent variables One such that gives equivalent form
II	Any transformation of dependent variable Four transformations of independent variables
III	Same as II

	Vertices of local pyramid of reference
I	x, x_u, x_v, x_{uu}, x_{uv}, x_{vv}, x_{uuu}, x_{vvv}
II	x, x_u, x_v, x_{uu}, x_{uv}, x_{vv}, x_{uuu}, x_{uvv}
III	x, x_u, x_v, x_{uu}, x_{uv}, x_{vv}, x_{uuu}, x_{uvv}, x_{uuuu}

	Hyperplanes with first order contact with S	Hyperplanes with second order contact with S
I	∞^4: $\sum_{i=4}^{8} \xi_i x_i = 0$	A pencil: $\xi_7 x_7 + \xi_8 x_8 = 0$
II	Same as I	Same as I
III	∞^5: $\sum_{i=4}^{9} \xi_i x_i = 0$	∞^2: $\xi_7 x_7 + \xi_8 x_8 + \xi_9 x_9 = 0$

	Hyperplanes with third order contact with S	Tangent line to C on S
I	None	$x_3 - \lambda x_2 = x_i = 0$, $(i = 4, \cdots, 8)$
II	None	Same as I
III	One: $x_9 = 0$	$x_3 - \lambda x_2 = x_i = 0$, $(i = 4, \cdots, 9)$

	Osculating S_2 to C	Locus of osculating S_2 to all C with same E_1
I	An S_2 determined by five hyperplanes	S_3
II	Same as I	S_3
III	An S_2 determined by six hyperplanes	S_3

	Locus of osculating S_2 to all C through P_x	Osculating S_3 to C
I	Cone of Del Pezzo V^2_4	An S_3 determined by four hyperplanes
II	Same as I	Same as I
III	Same as I	An S_3 determined by five hyperplanes

	Locus of osculating S_3 to all C with same E_2	Locus of osculating S_3 to all C with same E_1
I	S_4	S_5
II	S_4	S_5
III	S_4	S_5

	Locus of osculating S_3 to all C through P_x
I	Cone V^5_6 in S_7
II	Same as I
III	Cone V^5_6 in the S_7: $x_9 = 0$

	Osculating S_4 to C	Locus of osculating S_4 to all C with same E_3
I	An S_4 determined by three hyperplanes	S_5
II	Same as I	S_5
III_1 $X = 0$	Same as I in $x_9 = 0$	Same as II in $x_9 = 0$
III_2 $X \neq 0$	An S_4 determined by four hyperplanes	S_5

	Locus of osculating S_4 to all C with same E_2	Locus of osculating S_4 to all C with same E_1
I	S_6	Infinity of hyperplanes in ambient space S_7
II	S_6	Same as I
III_1	Same as II in $x_9 = 0$	Same as I in the S_7: $x_9 = 0$
III_2	S_6	Cone V^2_7 with fifth order contact with S

	Envelope of hyperplanes as λ' varies	Number of directions in which contact of this cone is of higher order than three
I	Cone V^2_6	Three: directions of double families of singular triples
II	Same as I	Four: directions of double families of singular triples
III_1	Same as I in $x_9 = 0$	Same as II in $x_9 = 0$
III_2	---	---

	Types of quasi-asymptotics
I	$y_{1,6}$ $y_{2,4}$
II	$y_{1,6}$ $y_{2,4}$
III	$y_{1,7}$ $y_{2,5}$ $y_{3,4}$
Fifth Canonical Form: $a' = 0$	Only quasi-asymptotics are v-curves which are of ten types.

VITA

Charles Franklin Bowles was born August 1, 1901 at Parkersburg, West Virginia. He received his early education in the public schools of that city, graduating from Parkersburg High School in June 1917. In the fall of 1918 he entered Marietta College, and was graduated with the degree of Bachelor of Arts in June 1922. During those years, under the instruction of Professor H. L. Coar, he received his first great interest in mathematics and his desire to devote his life to its study. The following year was spent in graduate study at Harvard University, from which institution he received the degree of Master of Arts in June 1923.

The following fall he went to the South Dakota State School of Mines as an instructor in mathematics, and has held a teaching position there since that time, later being promoted to the rank of assistant professor.

He entered the University of Chicago in June 1925 and was in residence during the summers of 1925 to 1929 inclusive and during the academic year 1929-1930, while on leave of absence from the School of Mines. During that time he studied under Professors Barnard, Bartky, Bell, Bliss, Coble, Lane, Logsdon, Lunn, Mitchell, and Murnaghan. He is indebted to Professors Logsdon and Lane for their instruction in his chosen field of geometry, and particularly to the latter, who suggested the subject of this paper, and whose unfailing sympathy, inspiration, and encouragement have made possible its preparation.